Magnetically-Coupled Rotors

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A system with intriguing oscillatory behavior can be created using small magnets fixed to the edges of parallel rotating structures. Under certain conditions, the two rotors will exchange velocities repeatedly in a manner similar to many coupled-oscillator systems, but without oscillation in position. This investigation began by examining the behavior of such an oscillator constructed with "Geomag" magnetic toys. In attempting to model the system, a simplified version of the problem was considered: two magnetic dipoles separated along a common axis of rotation. The model's predictions were compared to data collected from an equivalent experimental apparatus and match very well.

I. INTRODUCTION

Recently one of us received a "Geomag" kit as a gift. These kits consist of a number of magnetic rods and steel ball bearings. The bearings can be used as junctions between the rods, allowing multiple attachments at arbitrary angles so as to form geometric structures.[2] The magnets in these kits are strong enough to support fairly complex structures, even in tension, and one that particularly caught our attention was a hanging double rotor. (See Figure 1.)

The rotors are suspended at a single contact point between hardened steel bearings, and as a result the friction is very low. If the angular velocities of the two rotors are similar, they will alternately exchange angular velocity. This behavior is similar to that of typical coupled oscillators, but the "oscillations" are oscillations of velocity rather than of position.

II. EXPERIMENTAL DATA

The Geomag coupled rotor assembly whose behavior triggered this investigation consists of thirty separate magnetic dipoles. We chose to work with a somewhat more simple arrangement of two magnetic dipoles on a common axis. To measure the motion of the dipoles, we used two PASCO rotary motion sensors attached to a Vernier LabPro interface. We mounted a small brass flywheel to each rotary motion sensor to increase the rotational inertia and proportionally decrease the effect of friction in the bearings. The magnetic dipole was supplied by gluing a disk-shaped neodymium magnet on the axis of each sensor, with the magnetic moment perpendicular to the axis. Both sensors were then mounted on a stand so that the rotational axes were co-linear. (See figure 2.)

The experimental data obtained by this method (figure 3) shows the oscillatory behavior very well. For the data set shown, rotor 1 was given an initial spin and rotor 2 was initially at rest. The velocity oscillations of rotor 2 increase in magnitude as the angular velocity of rotor 1 decreases, until the system reaches the point where the two rotors have the same angular velocity. At that point, the two begin to alternately exchange velocities. The decay is consistent with damping from a constant frictional torque.[3]

III. THEORY

The field of a magnetic dipole in coordinate-free form is given by[1]

$$\vec{B} = \frac{\mu_o}{4\pi r^3}\left[3(\vec{m}\cdot\hat{r})\hat{r} - \vec{m}\right] . \tag{1}$$

The dipole moments are perpendicular to the common axis of rotation, so $\vec{m}\cdot\hat{r} = 0$ and the magnetic field from dipole 1 at the position of dipole 2 is

$$\vec{B}_1 = -\frac{\mu_o \vec{m}_1}{4\pi r^3} . \tag{2}$$

The magnetic torque on dipole 2 due to dipole 1 is

$$\vec{\tau}_2 = \vec{m}_2 \times \vec{B}_1 = -\frac{\mu_o}{4\pi r^3}(\vec{m}_2 \times \vec{m}_1) . \tag{3}$$

The two dipoles have the same magnitude $|\vec{m}|$, so the torque has magnitude

$$\tau_2 = -\frac{\mu_o}{4\pi r^3} m^2 \sin(\theta_1 - \theta_2) = I\ddot{\theta}_2 , \tag{4}$$

where I is the rotational inertia. The angular acceleration is then

$$\ddot{\theta}_2 = -\beta \sin(\theta_1 - \theta_2) \tag{5}$$

where

$$\beta \equiv \frac{\mu_o m^2}{4\pi r^3 I} . \tag{6}$$

For the purpose of comparing the model with the experimental apparatus, we add a constant frictional torque term $-b\dot{\theta}/|\dot{\theta}|$, and then write the equations of motion for each rotor:

$$\ddot{\theta}_1 = -\beta \sin(\theta_2 - \theta_1) - b\frac{\dot{\theta}_1}{|\dot{\theta}_1|} \tag{7}$$

$$\ddot{\theta}_2 = -\beta \sin(\theta_1 - \theta_2) - b\frac{\dot{\theta}_2}{|\dot{\theta}_2|} \,. \tag{8}$$

A closed-form solution to this set of equations is not available, but we can gain some insight into the problem by looking at the sum and difference of equations 7 and 8. Define

$$\mathcal{S} \equiv \theta_1 + \theta_2 \tag{9}$$

and

$$\mathcal{D} \equiv \theta_1 - \theta_2. \tag{10}$$

Adding equations 7 and 8 gives us

$$\ddot{\mathcal{S}} = -b\left(\frac{\dot{\theta}_1}{|\dot{\theta}_1|} + \frac{\dot{\theta}_2}{|\dot{\theta}_2|}\right) \tag{11}$$

and subtracting them gives us

$$\ddot{\mathcal{D}} = 2\beta \sin \mathcal{D} - b\left(\frac{\dot{\theta}_1}{|\dot{\theta}_1|} - \frac{\dot{\theta}_2}{|\dot{\theta}_2|}\right) \,. \tag{12}$$

Equation 12, with $b = 0$, is the equation for the simple pendulum, with the coordinate system rotated so that the equilibrium position is at π instead of at 0. This tells us that —whatever the behavior of the individual rotors— the *difference* between the two rotors behaves similarly to the physical pendulum. Equation 11, with $b = 0$, tells us that the total angular velocity —and thus the angular momentum— is conserved. With $b \neq 0$, we can see that the angular momentum decreases in an unexpected stepwise fashion: $\ddot{\mathcal{S}} = -2b$ if the signs of $\dot{\theta}_1$ and $\dot{\theta}_2$ are both positive, and $\ddot{\mathcal{S}} = 0$ if one of $\dot{\theta}_1$ or $\dot{\theta}_2$ is negative.

IV. SIMULATION

The equations of motion for this system lend themselves well to numeric solution. We used a fourth-order Runga-Kutta algorithm, with parameters β and b chosen to match our experimental data, to obtain the results shown in figure 4. The unexpected stepwise decrease in the total angular velocity predicted by equation 11 is clearly visible. Looking back at figure 3, one can see hints of these same steps in the sum, particularly around $t = 25$ seconds, although this is at the limits of our experimental resolution.

V. SUMMARY

When coupled oscillators are introduced in undergraduate physics courses, they are usually discussed in terms of oscillating position. In this system, there is no common restoring force and the oscillations are in velocity, rather than position. The overall behavior of the system contains elements that link it to the behavior of other well-known systems: damped simple harmonic motion and rotation with friction. There is a wealth of interesting behavior in the apparatus, and numerous conceptual links to other systems. For example, the factor-of-two change in period of each rotor at the crossover point is analogous to the change in period of a physical pendulum when it goes from "looping" around the axis to swinging back and forth. There is also an unexpected aspect of the theory —the stepwise decrease in total angular momentum— which appears to be real in the experimental data. Despite the complexity of the behavior, it can be modeled computationally without difficulty and the model matches the observed behavior closely.

Acknowledgments

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[1] David J. Griffiths, *Introduction to Electrodynamics* (Prentice-Hall, 1999), 3rd ed., 246.

[2] W. C. K. Poon, *Two magnets and a ball bearing: A simple demonstration of the method of images* Am. J. Phys. **71**, 943–947 (2003).

[3] John C. Simbach and Joseph Priest, *Another look at a damped physical pendulum* Am. J. Phys. **73**, 1079–1080 (2005).

FIG. 1: Geomag coupled-rotor configuration

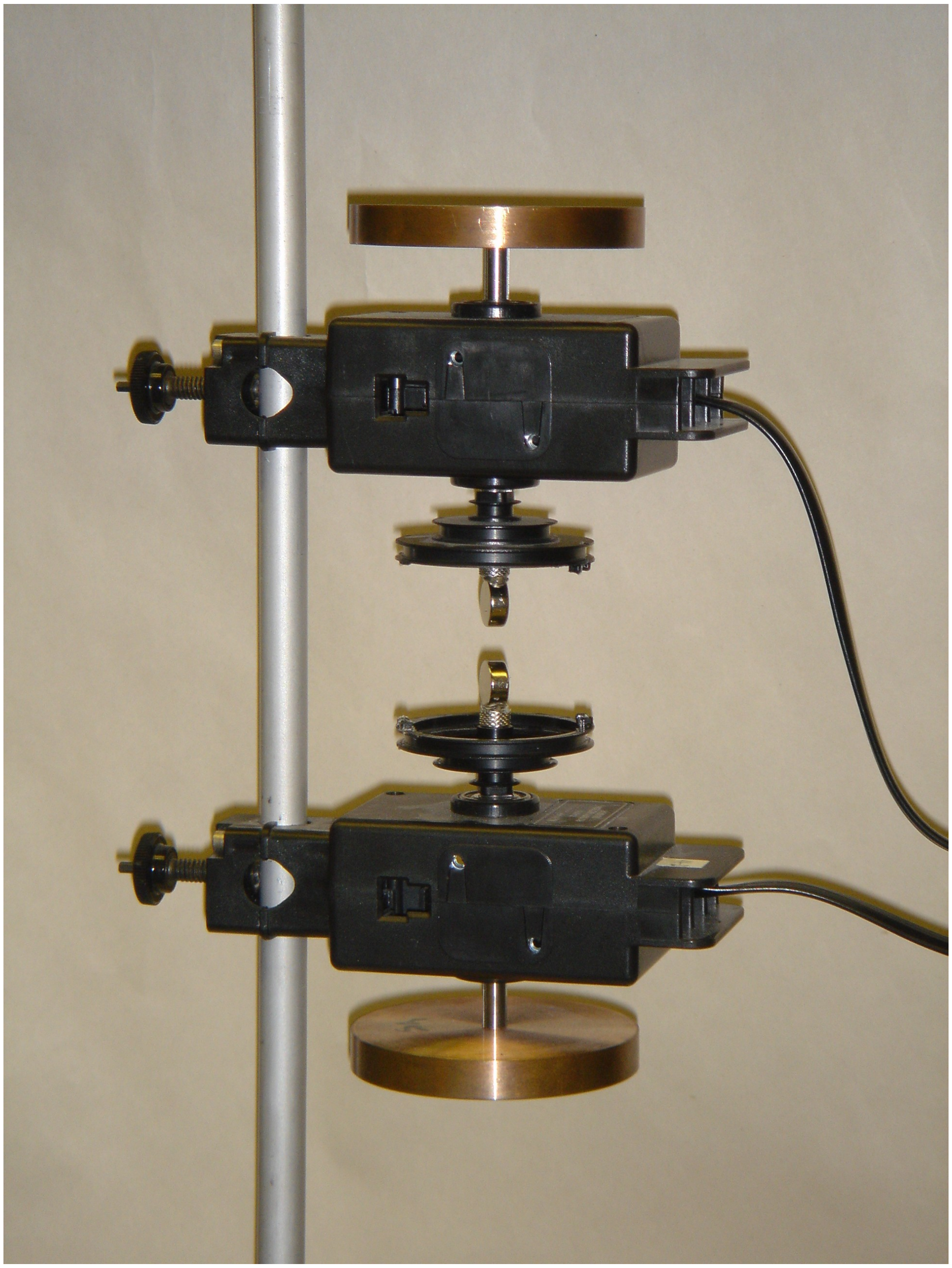

FIG. 2: Apparatus used for experimental observations. Adjusting the spacing between the two rotors affects the strength of the interactions, but does not qualitatively change the behavior.

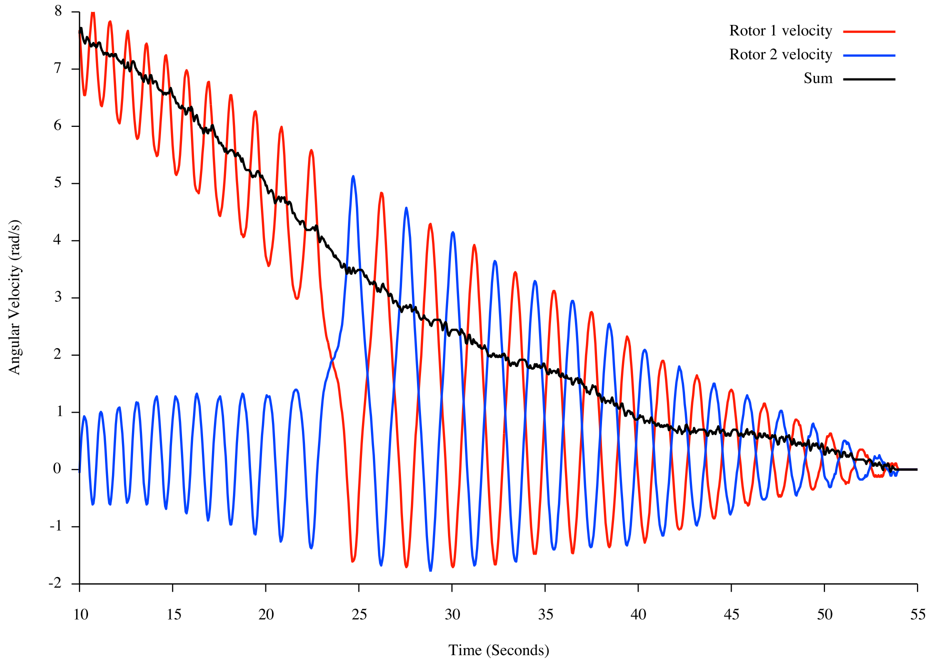

FIG. 3: Experimental data. Rotor 1 was given an initial angular velocity, rotor 2 was initially at rest.

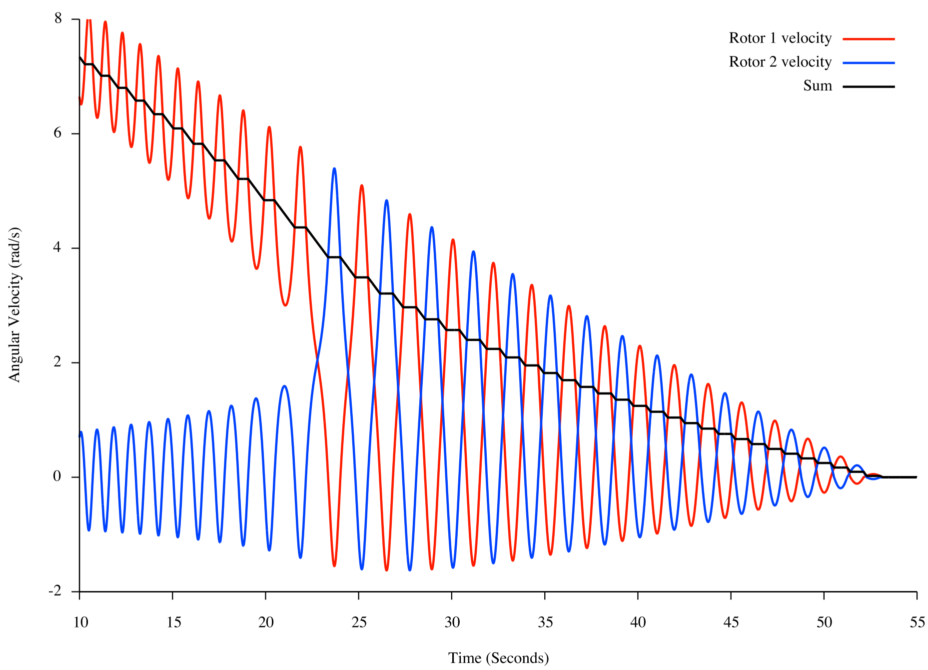

FIG. 4: Computed behavior of the system, showing angular velocity of both rotors and the total velocity.