Strange attractors of a bouncing ball

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We demonstrate the existence of strange attractors in a simple dissipative mechanical system, a bouncing ball subject to repeated impacts with a vibrating table.

I. INTRODUCTION

Chaos is an old term which has gained a new meaning in the scientific community during the past few years. Chaos theory studies the instabilities that arise in nonlinear systems. The instabilities are not the result of external noise, but are inherent within the deterministic equations describing the system's dynamics. It should come as no surprise that deterministic equations can produce random results; in fact, this is exactly how a computer's random number generator functions. What may come as a surprise is that equations long used in physics (e.g., Newton's equations) can produce seemingly random results when nonlinear terms are included. The irregular behavior of deterministic systems is sometimes called "deterministic chaos" to distinguish the term chaos from its more colloquial usage. An introduction to the field that includes an extensive collection of useful reprints has recently been published.[1]

Dissipative dynamical systems can exhibit a fascinating chaotic solution known as a strange attractor. An attractor of a dynamical system is simply where all the solutions go after a long time; it is the asymptotic orbit for a dissipative system. A strange attractor is loosely defined as anything that is not a simple attractor. Simple attractors are what we usually study in physics and consist of constant solutions (equilibrium or fixed points), periodic solutions (limit cycles), or quasiperiodic orbits (motion which is periodic in each variable). In this paper we describe a simple mechanical apparatus that allows us to see and hear the formation of a strange attractor in real-time.

Strange attractors that arise in mechanical systems have been studied by several researchers.[2] Most notable in terms of its experimental simplicity is the mechanical system constructed by Moon and Holmes consisting of a metal beam subject to periodic oscillations at one end and magnetic forces at the other.[3] In their mathematical model of the beam, Moon and Holmes assume that the beam oscillates only in its lowest-order mode. Despite this assumption, they manage to obtain good qualitative agreement between the strange attractor that arises in the modeling equations and the strange attractor from the experimental system. Still, in order to explore the connection between chaos in equations and chaos in experimental systems (e.g., the role played by noise) it would be advantageous to work with a system where the correspondence between the experimental apparatus and the mathematical model is assured.

Such a mechanical system has been recently discussed in this Journal.[4] The model consists of a ball that is free to bounce inelastically on a table which moves sinusoidally up and down (Fig. 1). An experimental system to study the repeated impacts of a ball with a sinusoidally vibrating table is simple to construct from a loudspeaker and a ball bearing. The bouncing ball system appears to have been studied theoretically for the first time by Holmes,[5] who showed the existence of periodic and chaotic motion (i.e., strange attractors) for suitable parameter values and initial conditions.

In this paper we show how to explore the chaotic motions of a bouncing ball by experimentally constructing a surface of section map. The simplicity of the system recommends itself to study by both theoreticians and teachers. The latter will especially find it helpful in explaining the basic concepts of nonlinear dynamics.

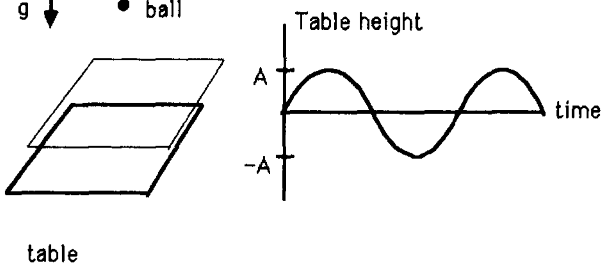

Fig. 1. Bouncing ball. A ball is free to bounce on a table which moves sinusoidally up and down.

II. STRANGE ATTRACTORS AND IMPACT MAPS

A Poincaré surface of section map is a convenient and useful technique through which to view the periodic and chaotic motions of a dynamical system.[6] A more natural map to consider in the bouncing ball system is the so-called "impact map." This map is equivalent to a surface of section map but is easier to obtain experimentally. The impact map is defined by recording the phase of the periodic forcing and relative force (or possibly velocity) of each impact between the ball and the table. If T is the period of the oscillating table, then the horizontal axis is the time modulo T. The vertical axis indicates the impact force between the ball and the table. At each collision, a dot is placed on the plot indicating a given impact's strength and phase.

The initial condition for an orbit of the bouncing ball consists of specifying the strength and phase of the first impact. The future evolution of the orbit is represented by the unique sequence of impact dots generated by a particu-

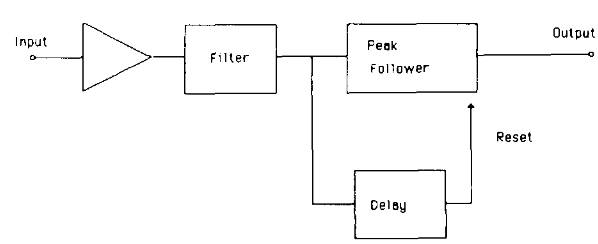

Fig. 2. Block diagram of impact map circuit.

lar initial condition. The set of all initial conditions of the impact map defines a certain dissipative (area contracting) mapping of a cylinder (since the horizontal axis is topologically a circle) onto itself.

The "attractor" (i.e., the asymptotic limit set) for a given collection of initial conditions can be exceedingly simple, or complex. For some initial conditions and parameter values the attacting set will simply be a collection of separate dots showing periodic motion. The number of distinct dots indicates the period relative to the forcing frequency.

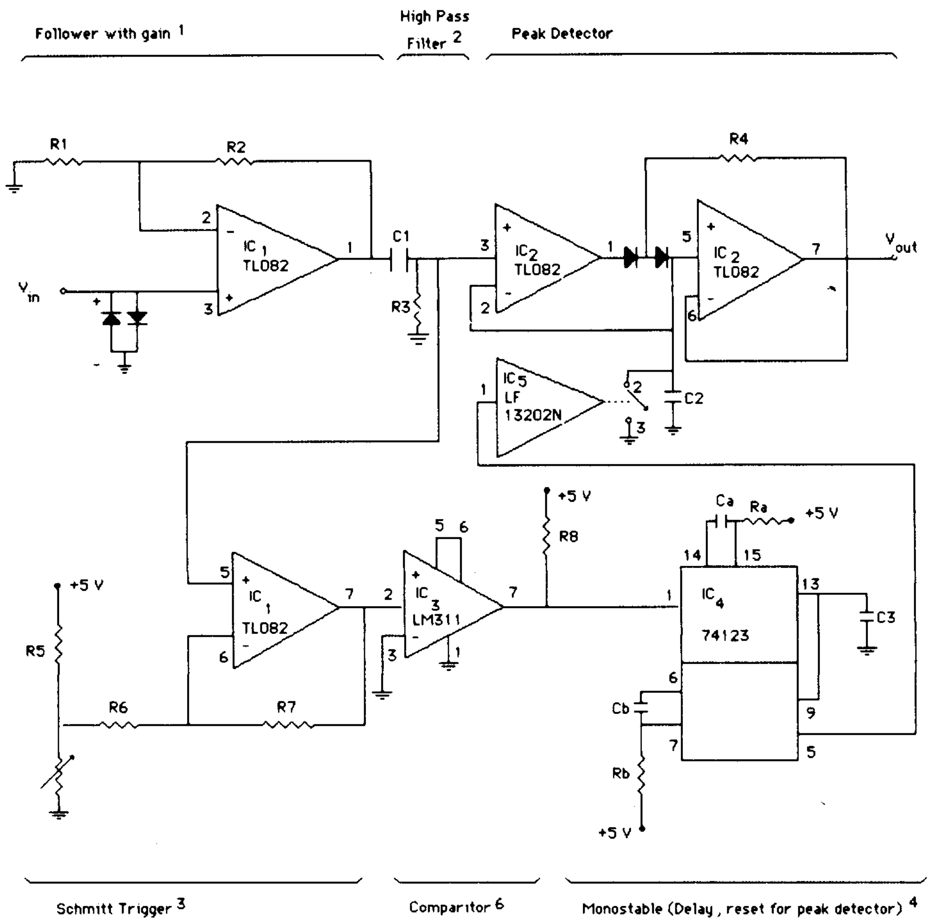

Fig. 3. Impact map circuit.

1. Gain of follower $G = (R1 + R2)/R1 = 20$.
2. High-pass filter, the cutoff frequency is $1/C_1R_3$.
3. The variable resistor (trimpot) is used to adjust the trigger level.
4. RaCa provide a 50-μs delay (for A to D conversion); RbCb send a 5-μs pulse to reset the holding capacitor or the peak detector.
5. Power connections:

$1C_1$,$1C_2$(TL082)	Pin 4	−15 V
op amp	Pin 8	+15 V
$1C_3$(Lm311)	Pin 4	−15 V
Comparator	Pin 8	+15 V
$1C_4$(SN74123)	Pin 8	GND
Dual monostable	Pin 16	+5 V
$1C_5$(LF13202N)	Pin 4	−15 V
	Pin 5	GND
	Pin 13	+15 V

6. The comparator output is a logic level signal coincident with the peak (bounce).
7. Resistor and capacitor values:

R1	0.2 kΩ	C1	0.01 μF
R2	3.8 kΩ	C2	0.01 μF
R3	22 kΩ	C3	0.1 μF
R4	47 kΩ	Ca	0.1 μF
R5	22 kΩ	Cb	0.01 μF
R6	1 kΩ		
R7	100 kΩ		
R8	470 Ω		
Ra8	1 kΩ		
Rb	1 kΩ		

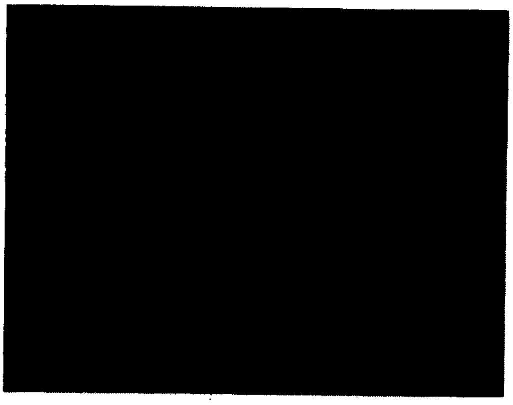

Fig. 4. Experimental impact map. A period two orbit of the bouncing ball system as seen on the storage oscilloscope.

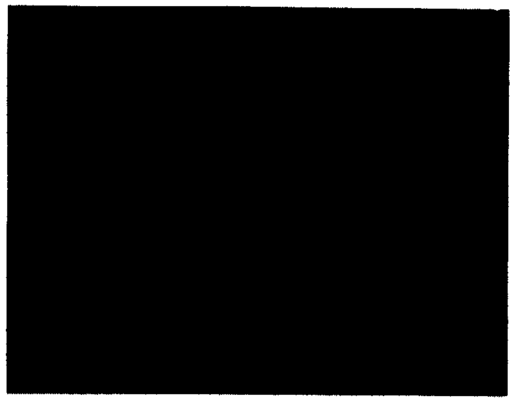

Fig. 6. The impact map of a strange attractor that arises immediately after the period four orbit.

For instance, two distinct dots indicates a period two orbit.

Yet other sets of initial conditions and parameter values lead to very complicated limit sets known as "strange attractors" whose mathematical and physical properties are still not fully understood.[7] Roughly, a strange attractor can be thought of as an infinitely long manifold (curve, surface, volume, etc., depending on the dimension of the underlying phase space) which is crumpled up in a bounded region of phase space. Loosely speaking, a strange attractor is akin to a space filling curve that fills out a limited section of phase space. Strange attractors are characterized by an experimentally measurable noninteger "fractal" dimensionality and appear to be self-similar over many length scales.[8] More important, nearby orbits on a strange attractor tend to diverge at an exponential rate.[9] This "sensitive dependence on initial conditions" is the chief definition of "chaos" (i.e., the random behavior of deterministic systems) and suggests that statistical methods are appropriate for modeling the dynamics of physical systems in the chaotic regime. Strange attractors and chaos are every bit as generic in nonlinear systems as stationary, periodic, or quasiperiodic behavior.

III. IMPACT MAP CIRCUIT

The experimental realization of the bouncing ball system has been described in a recent article in this Journal.[4] A speaker driven by a function generator serves as the vibrating table. A small steel ball bounces against a concave lens glued to the speaker. Fastened to the top surface of the lens

Fig. 5. The impact map for a period four orbit.

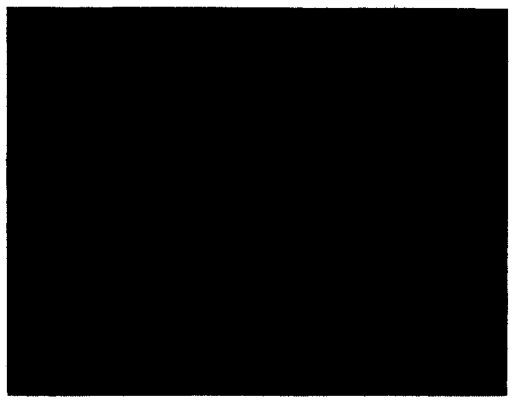

Fig. 7. The development with increasing forcing amplitude of the strange attractor is seen in Figs. 6–9.

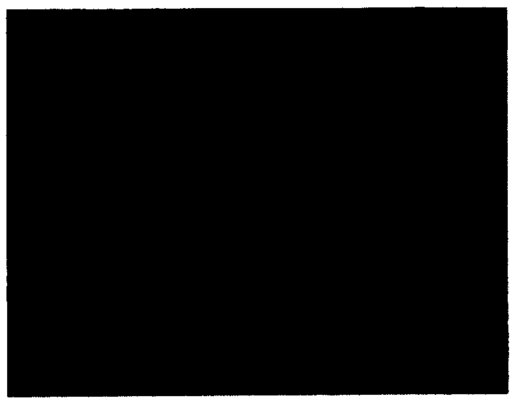

Fig. 8. The strange attractor grows downward as the forcing amplitude is increased.

Fig. 9. The strange attractor eventually fills out a section of the sine curve.

Fig. 10. Using a smaller ball allows the leaves of the strange attractor to be distinguished more readily.

Fig. 11. The strange attractor shown here is obtained at a large forcing amplitude and is almost identical to those obtained by Holmes from computer simulations.

is a thin piezoelectric film. Every time the ball hits the lens the film generates a voltage spike. Examples of the piezoelectric film output are shown in Ref. 4.

In order to construct an experimental impact map we record the strength and phase of each collision with the piezoelectric film on a storage oscilloscope. The horizontal sweep of the oscilloscope can be generated by a ramp voltage obtained from the function generator and is proportional to the forcing frequency. Alternatively, a quicker method to set up the horizontal axis is to drive the horizontal sweep with a time base while "triggering" off the function generator signal which is powering the speaker. This method also has the advantage of allowing us to adjust the phase viewed on the storage oscilloscope to any desired interval.

A bit more circuitry is required to plot the strength and phase of a given impact. To begin with, the output signal from the piezoelectric film must be amplified and filtered to remove low-frequency components such as the vibration of the speaker. Next, a peak detector with follower is employed to mark an individual bounce. Once detected, the peak voltage is held by a variable delay which allows us to change the size of the dot seen on the storage oscilloscope. The basic elements of the impact map circuit are shown in Fig. 2. The full circuit, as shown in Fig. 3, is also ideally suited for A/D conversion, thereby allowing us to record the evolution of an orbit on a microcomputer as well as a storage oscilloscope.

IV. EXPERIMENTAL STRANGE ATTRACTORS

Photographs showing both periodic and chaotic motion were taken from the storage oscilloscope. When the forcing amplitude of the speaker is increased (for a fixed frequency) the bouncing ball system undergoes a period doubling cascade that culminates in chaotic motion on a strange attractor.[4] Figures 4 and 5 show period two and period four orbits, respectively. The period four orbit exists right before the onset of the strange attractor shown in Fig. 6. Figures 6–9 detail the development of the strange attractor as the forcing amplitude is increased. The phase on the horizontal axis goes from 0 to T. The strange attractor in Figs. 6–9 closely resembles a sine wave. However, closer examination will reveal that the strange attractor is composed of many leaves. On the left-hand side of the strange attractor we can clearly distinguish two separate leaves. In principal, if we could expand our view of the strange attractor we would see new leaves appearing at ever finer length scales. However, the noise inherent in an experimental system destroys this delicate structure so that it no longer exists below a noise limited length scale.

We also note that the branches of the strange attractor grow downward as the forcing amplitude is increased. The growth of the strange attractor as we enter farther into the chaotic regime presumably corresponds with an increase in its fractal dimensionality. To test this hypothesis, we are currently digitizing our data in order to calculate the fractial dimensionality of the strange attractor.

In Figs. 6–9 we used a 3/8-in. steel ball. For Figs. 10 and 11 we used a 1/4-in. steel ball. In addition, the phase interval viewed is varied to help focus in on different sections of the attractor. Figures 10 and 11 allow us to clearly distinguish different sheets of the strange attractor. Holmes, in an earlier numerical study, developed a simplified model of the bouncing ball system. The strange attractor presented by Holmes from computer simulations is almost identical with the strange attractor in Fig. 11 from our experimental system.[5]

V. CONCLUSION

We have demonstrated the existence of a strange attractor in an impact oscillator. Our results are in excellent qualitative agreement with the previous theoretical work of Holmes.[5] Currently, we are engaged in a detailed quantitative comparison between computer simulations and the actual dynamics of a bouncing ball subject to repeated impacts with a vibrating table.

The apparatus described herein is useful for demonstrating the "structure" inherent within chaotic systems. In the chaotic regime, the sound produced by a sequence of impacts appears random to the listener. However, the experimental impact map shows that there is, in fact, a strong correlation between the phase and force of an impact.

Our circuit description should prove useful in the analog and digital analysis of any dynamical system which is subject to a relatively low frequency of forcing. Moreover, this particular system provides a dramatic and fascinating illustration of a strange attractor and is pedagogically useful in demonstrating the complex dynamics inherent within the simplest nonlinear oscillators.

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Graphical methods in geometric and Gaussian optics

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Graphical methods showing the relationships among object and image positions, focal length and magnification for geometric optics are extended to the use of Newtonian variables. By defining similar quantities for Gaussian beams, we develop graphical methods to describe the relationships among the parameters specifying Gaussian beams. Examples are included.

INTRODUCTION

Graphical methods are useful in understanding the relationship among object position, image position, focal length, and magnification in simple optical systems using lenses or mirrors. Two methods for geometric optics have been reported. It is the purpose of the present work to review these methods and to extend them to Newtonian variables for use in more rigorous introductory optics courses. Also, following a brief discussion of Gaussian beams, the methods are adapted to the optics of these beams for more advanced work.

GRAPHICAL METHODS FOR GEOMETRIC BEAMS

The graphical method used by Halliday and Resnick[1] and discussed by Bartlett[2] used the normalized quantities $p/|f|$ and $q/|f|$, where p and q are the object and image distances and f is the focal length of a lens or mirror. These quantities are plotted as abscissa and ordinate along with the hyperbola $|f|/p + |f|/q = 1$ (for positive elements) or -1 (for negative elements). Since the coordinates of any point on the hyperbola are $p/|f|$ and $q/|f|$, one of these quantities can be read directly if the other is known. Selected points on the hyperbola can be numbered to represent the transverse geometric magnification, $m_g = -q/p$. For completeness it is noted that the longitudinal magnification is $-m_g^2$.

Translation of the center of the coordinate system permits the use of the variables x/f and x'/f, where $x = p - f$, $x' = q - f$, and $xx' = f^2$, quantities used in the Newtonian form of the lens equation. The hyperbola then becomes equilateral. It must be understood that for positive elements, x is measured from the front focal point and x' is measured from the back focal point, while for negative elements, x is measured from the back focal point and x' from the front focal point. The geometric magnification can be expressed as

$$m_g = -f/x = -x'/f = -\operatorname{sgn}(x/f)\sqrt{(x'/x)}, \quad (1)$$

where $\operatorname{sgn}(x/f)$ represents the algebraic sign of (x/f). Equation (1) shows that x'/f is the reciprocal of x/f and so must have the same sign. Also, the magnification is the negative of x'/f, the ordinate of a graph of x'/f vs x/f. These results illustrate the power and importance of the Newtonian form of the lens equation in that the magnitude and sign of x'/f and m_g are easily determined.

Figure 1 shows the adaptation of Halliday and Resnick's method to be the variables x/f and x'/f. Only one plot is required since the value of f rather than the absolute value of f is used. Corresponding values of x and x', or p and q can be easily found and the magnification read directly as $-x'/f$. A drawback of plotting x'/f vs x/f is that it blurs the distinction between real and virtual objects and images; real objects and images occur for x and $x' > -f$, or for x/f and $x'/f > (-1)\operatorname{sgn}(x/f)$.

Another graphical method was presented by Wilson[3] and discussed by Horsfield.[4] It has the advantages of using