Some simple chaotic jerk functions

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A numerical examination of third-order, one-dimensional, autonomous, ordinary differential equations with quadratic and cubic nonlinearities has uncovered a number of algebraically simple equations involving time-dependent accelerations (jerks) that have chaotic solutions. Properties of some of these systems are described, and suggestions are given for further study.

I. INTRODUCTION

One of the most remarkable recent developments in classical physics has been the realization that simple nonlinear deterministic equations can have unpredictable (chaotic) long-term solutions. Chaos is now thought to be rather common in nature, and the study of nonlinear dynamics has brought new excitement to one of the oldest fields of science. The widespread availability of inexpensive personal computers has brought many new investigators to the subject, and important research problems are now readily accessible to undergraduates. An interesting and yet unsolved problem is to determine the minimum conditions necessary for chaos. This paper will describe several examples of chaotic flows that are algebraically simpler than any previously reported and will suggest further lines of promising investigation.

The chaotic system to which one is usually first introduced is the logistic equation,[1]

$$x_{n+1}=Ax_n(1-x_n), \tag{1}$$

which is remarkably simple and yet exhibits many of the common features of chaos. For most values of A in the range 3.5699... to 4, it produces a sequence of x values that exhibit sensitive dependence on initial conditions and long-term unpredictability. Its behavior can be studied with a simple computer program or even a pocket calculator.

Equation (1) is a one-dimensional iterated map in which the variable x advances in discrete time steps or jumps. Most of the equations of physics, and science in general, are more naturally expressed in the form of differential equations in which the variables evolve continuously in time. Newton's second law is the prototypical example of such a continuous dynamical process.

Whereas chaos can arise in discrete-time systems with only a single variable, at least three variables are required for chaos in continuous-time systems.[2] The reason is that the trajectory has to be nonperiodic and bounded to some finite region, and yet it cannot intersect itself because every point has a unique direction of flow. Newton's second law in one dimension (1D) inherently contains two variables because it involves a second derivative. It is really two equations, a kinematic one defining the velocity, $dx/dt=v$, and a dynamic one describing the rate of change of this velocity, $dv/dt=F/m$. Thus Newton's second law in 1D with a force that depends only on position and velocity cannot produce chaos since there are only two phase-space variables (x and v).

In two spatial dimensions, there are four phase-space variables, and thus chaos is possible. For example, a planet orbiting a single massive star is described by two spatial components and two velocity components since the orbit lies in a plane. The equation of motion is nonlinear because the force is proportional to the inverse square of the separation. However, this system does not exhibit chaos because there are two constants of the motion—mechanical energy and angular momentum—which reduce the phase-space dimension from four to two. With a third object, such as a second star, the planet's motion can be chaotic, even when the motions are coplanar, because the force on the planet is no longer central, and its angular momentum is thus not conserved. The three-body problem was known 100 years ago to have chaotic solutions and has never been solved in the sense of deriving an analytic expression for the position of the bodies as a function of time.

A one-dimensional system can exhibit chaos if the force has an explicit time dependence. For example, a sinusoidally driven mass on a nonlinear spring with a cubic restoring force ($-kx^3$) and linear damping ($-b\dot{x}$) obeys an equation

$$\ddot{x}+b\dot{x}+kx^3=A\sin\omega t, \tag{2}$$

where $\dot{x}\equiv dx/dt$. This is a special case of Duffing's equation[3] whose chaotic behavior has been studied by Ueda.[4] It is a useful model for any symmetric oscillator such as a mass on a spring driven to a sufficiently large amplitude that the restoring force is no longer linear. Note that x and t can be rescaled so as to eliminate two of the four parameters (b, k, A, and ω). For example, we can take $k=\omega=1$ without loss of generality. Thus the behavior of the system is determined entirely by two parameters (b and A in this case), and by the initial conditions, $x(0)$ and $\dot{x}(0)$. Equation (2) is known to have chaotic solutions[5] for $b=0.05$ and $A=7.5$, among other values.

Systems such as Eq. (2) with an explicit time dependence can be rewritten in autonomous form (t does not appear explicitly) by defining a new variable $\phi=\omega t$, leading to a system of three, first-order, ordinary differential equations (ODEs) such as

$$\dot{x}=v, \quad \dot{v}=-bv-kx^3+A\sin\phi, \quad \dot{\phi}=\omega. \tag{3}$$

The new variable ϕ is a periodic phase, and thus the global topology of the system is a torus. Other standard examples of chaotic autonomous ODEs with three variables include the Lorenz[6] and Rössler[7] attractors, which have only quadratic nonlinearities, but each of which has a total of seven terms on its right-hand side.

An earlier paper[8] described a computer search that revealed 19 examples of chaotic flows that are algebraically simpler than the Lorenz and Rössler systems. These autonomous equations all have three variables (x, y, and z) and either six terms and one quadratic nonlinearity or five terms and two quadratic nonlinearities.

II. JERK FUNCTIONS

Recently Gottlieb[9] pointed out that the simplest ODE in a single variable that can exhibit chaos is third order, and he suggested searching for chaotic systems of the form $\dddot{x}=j(x,\dot{x},\ddot{x})$, where j is a jerk function (time derivative of acceleration).[10,11] He showed that case A in Ref. 8,

$$\dot{x}=y, \quad \dot{y}=-x+yz, \quad \dot{z}=1-y^2, \tag{4}$$

can be recast into the form

$$\dddot{x}=-\dot{x}^3+\ddot{x}(x+\ddot{x})/\dot{x}, \tag{5}$$

and he wondered whether yet simpler forms of the jerk function exist that lead to chaos. Equation (4) is a special case of the Nosé–Hoover thermostated dynamic system,[12,13] which exhibits time-reversible Hamiltonian chaos.[14]

Careful examination of the other 18 equations in Ref. 8 shows that most if not all of them can be reformulated into a third-order ODE in a single variable. This exercise is well suited for a student with an elementary knowledge of differential calculus. For example, case I in Ref. 8 can be written as

$$\dddot{x}+\ddot{x}+0.2\dot{x}+5\dot{x}^2+0.4x=0, \tag{6}$$

which in some ways is more appealing than Eq. (5) since it has only polynomial terms and a single quadratic nonlinearity.

It is interesting to ask under what conditions a system of m first-order ODEs in m variables can be written as an mth order ODE in a single variable. A theorem by Takens[15] assures us that almost any variable from an m-dimensional system can be used to reconstruct the dynamics provided a sufficient number of additional variables are constructed from the original variable by successive time delays, $x(t)$,$x(t-\tau)$,$x(t-2\tau)$,$x(t-3\tau)$,..., but it may require as many as $2m+1$ such time lags (called the ''embedding dimension''). This condition was subsequently relaxed to $2m$.[16] It is reasonable to assume that a differential equation of order $2m$ would also suffice. However, if we are free to choose the variable optimally, there is reason to hope that an embedding of m might suffice in most cases.

Systems for which this is apparently not the case include ones with periodic forcing such as Eq. (3). The reason is that one of the variables, ϕ, lies on a circle that requires a Euclidean dimension of 2 to embed it. If we replace the sin ωt in Eq. (2) with a new variable y that obeys the harmonic oscillator equation, $\ddot{y}=-y$, Eq. (3) with $k=\omega=1$ can be written as a fourth-order ODE,

$$\ddddot{x}+b\dddot{x}+\ddot{x}+3x^2\ddot{x}+b\dot{x}+6x\dot{x}+x^3=0. \tag{7}$$

It is interesting to note that the trigonometric nonlinearity (sin ϕ) in Eq. (3) has been replaced exactly by a small number of polynomial terms, suggesting that equations such as Eq. (7) are rather more general than would at first appear.

The term $\ddddot{x}\equiv d^4x/dt^4$ is the time derivative of the jerk, which might be called a ''spasm.'' It has also been called a ''jounce,'' a ''sprite,'' a ''surge,'' or a ''snap,'' with its successive derivatives, ''crackle'' and ''pop.''[17] Note that the parameter A does not appear in Eq. (7), but it does appear in the initial conditions, whose number is one greater than in Eq. (3). The initial conditions must satisfy the constraint $A=\dddot{x}(0)+b\ddot{x}(0)+3x^2(0)\dot{x}(0)$ for Eq. (7) to be equivalent to Eq. (3).

In one sense, equations involving the jerk ($\dddot{x}$) and its derivatives are unremarkable. Newton's second law ($\ddot{x}=F/m$) leads necessarily to a jerk whenever the force F in the frame of reference of the mass m has a time dependence, either explicitly $[F(t)]$ or implicitly $[F(x,\dot{x})]$. Except in a few special cases such as a projectile moving without drag in a uniform gravitational field, nonzero jerks exist. For example, a mass on a linear spring has a sinusoidally varying jerk, as well as all higher derivatives. However, in such a case, only two phase-space variables (x and $\dot{x}$) are required to describe the motion. In the cases considered here, not only is the jerk nonzero, but the acceleration ($\ddot{x}$) is an independent phase-space variable necessary to describe the motion. Such an equation might arise naturally in a case like a planet orbiting a pair of fixed massive stars where there are two spatial dimensions (and thus four phase-space variables) but with the trajectory constrained to a three-dimensional subset of the phase space by the conservation of energy.

III. NUMERICAL SEARCH PROCEDURE

Since Eq. (6) demonstrates the existence of chaotic jerk functions with only quadratic nonlinearities, it is interesting to identify the simplest such function. The most general second-degree polynomial jerk function is

$$\begin{aligned} j=&(a_1+a_2x+a_3\dot{x}+a_4\ddot{x})\ddot{x}+(a_5+a_6x+a_7\dot{x})\dot{x} \\ &+(a_8+a_9x)x+a_{10}, \end{aligned} \tag{8}$$

for which the goal is to find chaotic solutions with the fewest nonzero coefficients and with the fewest nonlinearities. Equation (6) ensures us that chaotic jerk functions with four terms and one nonlinearity exist. Thus we seek cases with four or fewer terms and one nonlinearity or fewer than four terms and two nonlinearities. Such cases would be at least as simple as the 19 cases in Ref. 8.

The numerical procedure was to choose randomly three or four of the coefficients (a_1 through a_{10}), set them to uniformly random values in the range -5 to 5, and then calculate the trajectory for randomly chosen initial conditions (x, $\dot{x}$, and $\ddot{x}$) in the range -5 to 5. The range -5 to 5 is arbitrary and poses no significant restriction because x and t can be rescaled. A fourth-order Runge–Kutta integrator with a step size of Δt=0.05 was used. The process was repeated the order of 10^7 times. The most common dynamic was for the trajectory to escape to infinity, and this was detected by stopping the calculation whenever $|x|+|\dot{x}|+|\ddot{x}|$ exceeded 10^4. Because of the quadratic nonlinearity, unbounded cases are usually identified within a few dozen iterations. The remaining solutions most often settled to a fixed point or limit cycle. Rare cases (the order of one in 10^4) exhibited chaotic solutions. Thus, in some sense, it is reasonable to conclude that chaos is relatively rare in algebraically simple systems of ODEs.[18]

The simplest way to detect chaos is to use its characteristic sensitive dependence on initial conditions. The calculation could be done twice in parallel with initial conditions that differ by a small ϵ_0. The quantity ϵ_0 can be chosen in (almost) any direction and assigned a value $\epsilon_0 \ll 1$ but several orders of magnitude greater than the computational precision. This is best done after a few thousand iterations to let the orbit converge to the attractor and to avoid unnecessary calculations for unbounded cases. The signature of chaos is that the separation of the orbits usually reaches a value of order unity ($\epsilon_0 \gtrsim 0.1$) quickly.

A more careful procedure and the one used here is to calculate the Lyapunov exponent.[19] This was done in a manner similar to that described above, except that after each iteration, the new orbit separation ϵ_1 was determined, and the separation was readjusted to ϵ_0 along the direction of ϵ_1. The largest Lyapunov exponent was then determined by averaging the natural logarithm of ϵ_1/ϵ_0 along the orbit. A decidedly positive Lyapunov exponent is a signature of chaos. Since there is always a zero Lyapunov exponent for a periodic flow, corresponding to a direction parallel to the flow, the search condition was for a Lyapunov exponent that remains in excess of 0.01 for 10^5 iterations. Typically about one chaotic solution emerged per hour of computing on a 66-MHz 486 personal computer. All the candidate chaotic cases found in this way were then tested with a smaller iteration step size ($\Delta t = 0.01$) for at least 10^6 iterations.

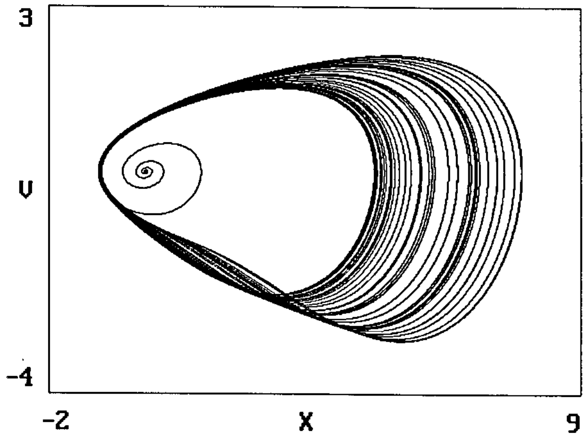

Fig. 1. Strange attractor for Eq. (9), with $A = 2.017$.

IV. SEARCH RESULTS

Chaotic flows in three dimensions (3D) can be characterized as either dissipative or conservative, according to whether the trajectory is attracted to a region of space with fractal dimension less than 3, a so-called strange attractor.[20] Dissipative systems have this property, and the attractor is independent of the initial conditions provided they lie in the basin of attraction. By contrast, a conservative system has a trajectory whose dimension depends on initial conditions, and is three for a chaotic trajectory, two for a quasiperiodic trajectory (two incommensurate frequencies), one for a periodic trajectory, and zero for an equilibrium point.

Three-dimensional chaotic flows must have one positive Lyapunov exponent, one zero exponent, and one negative exponent. The sum of the exponents is the rate of volume expansion for a cluster of initial conditions. This sum cannot be positive for bounded trajectories. If it is negative (dissipative), the initial conditions are drawn to an attractor whose volume is zero because its dimension is less than three (just as a 2-D surface has zero volume). If the sum of the exponents is zero (conservative), there is no contraction, and the chaotic trajectory fills some 3-D region, perhaps with a fractal boundary.

It is relatively easy to calculate numerically the rate of volume expansion, and from that, the negative Lyapunov exponent for a chaotic jerk function. It is given in terms of the sum of the Lyapunov exponents by $V^{-1}\, dV/dt = \Sigma L = \partial j / \partial \ddot{x} = a_1 + a_2 x + a_3 \dot{x} + 2 a_4 \ddot{x}$. Since this expression depends on x and its derivatives, in general it must be averaged along the trajectory. The dimension can then be estimated using the Kaplan–Yorke conjecture,[21] $D_{\mathrm{KY}} = 2 - L_1/L_3$, where L_1 is the positive exponent and L_3 is the negative exponent. The exponent L_2 is zero. Dissipative systems are usually easier to identify because they are less sensitive to initial conditions, have lower dimension, and are more robust to errors in the numerical method. They are also more representative of real physical systems since dissipation is nearly always present in some degree. These systems will be discussed first.

A. Dissipative systems

The simplest chaotic dissipative system that was found has all its coefficients equal to zero except a_1, a_7, and a_8. Two of these coefficients can be set to unity without loss of generality, and the remaining coefficient was arbitrarily taken as $a_1 \equiv -A$, leading to the equation,

$$\dddot{x} + A\ddot{x} - \dot{x}^2 + x = 0. \tag{9}$$

It is unlikely that any algebraically simpler form of an autonomous chaotic flow exists because the above equation has the minimum number of terms that allows an adjustable parameter and it has only a single quadratic nonlinearity. It can be equivalently written as three, first-order, ordinary differential equations with a total of five terms,

$$\dot{x} = v, \quad \dot{v} = a, \quad \dot{a} = -Aa + v^2 - x. \tag{10}$$

This is one fewer term or nonlinearity than in any of the 19 cases previously found and two fewer than in the Rössler equations. This case has been described in detail elsewhere.[22] It is a special case of Eq. (6) with the $\dot{x}$ term absent. It was presumably not discovered previously because the range of A for which chaos occurs is very narrow.

Equation (9) has bounded solutions for $2.017... < A < 2.082...$, a period-doubling route to chaos, and a nearly parabolic return map that strongly resembles the logistic equation. The positive Lyapunov exponent is largest for $A \cong 2.017$, and the exponents (base-e) at that value are $L \cong 0.0550$, 0, -2.0720, corresponding to a Kaplan–Yorke dimension of 2.0265. The attractor is approximately a Möbius strip, and the basin of attraction is shaped like a tadpole with a tail that apparently extends to infinity along the $-a$ axis.[23] Figure 1 shows a projection of the attractor onto the $x-v$ plane including a portion of the trajectory as it spirals outward to the attractor from an initial condition near (but not at!) the unstable saddle-focus at the origin. The eigenvalues, given by the characteristic equation, $\lambda^3 + A\lambda^2 + 1 = 0$, are within about 1% of $\lambda \cong -2.24$, $0.1 \pm 0.66i$ over the range of A for which bounded solutions exist.

A similar example of a chaotic flow was found in which the $\dot{x}^2$ term is replaced with $x\dot{x}$,

$$\dddot{x} + A\ddot{x} - x\dot{x} + x = 0, \tag{11}$$

but this case is equivalent to Eq. (9) to within a constant as can be seen by differentiating Eq. (9) with respect to time and defining a new variable $v \equiv \dot{x}$. It is chaotic over the same range of A as is Eq. (9). Figure 2 shows its attractor for $A = 2.017$ projected onto the $x-v$ plane including a portion

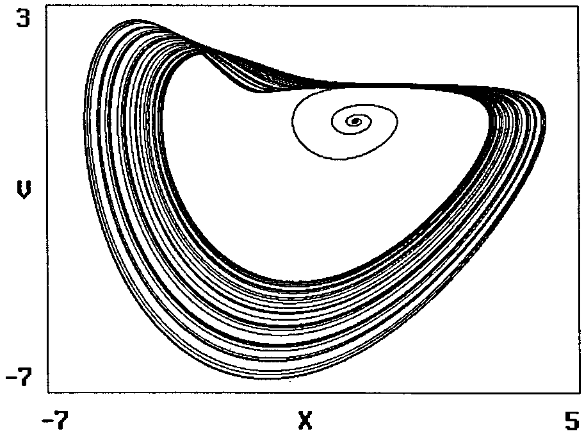

Fig. 2. Strange attractor for Eq. (11), with $A=2.017$.

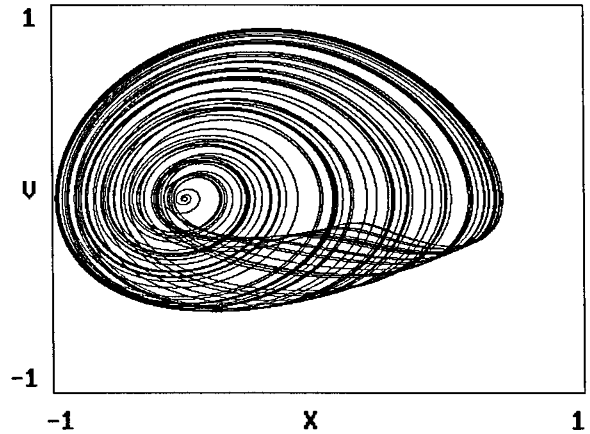

Fig. 3. Strange attractor for Eq. (15), with $A=0.5$ and $B=0.25$.

of the trajectory as it spirals outward to the attractor from an initial condition near (but not at!) the unstable saddle focus at the origin. These appear to be the only two examples of dissipative chaotic jerk equations with three terms and one quadratic nonlinearity.

Two cases were found with three terms and two quadratic nonlinearities, a form analogous to Eq. (9):

$$\dddot{x}+Ax\ddot{x}-\dot{x}^2+x=0; \tag{12}$$

and one analogous to Eq. (11):

$$\dddot{x}+Ax\ddot{x}-x\dot{x}+x=0. \tag{13}$$

Equation (12) has an attractor that strongly resembles Fig. 1 for $A=0.645$, and Eq. (13) has an attractor that resembles Fig. 2 for $A=-0.113$, although the initial conditions must be chosen carefully since their basins of attraction are relatively small.

Eight functionally distinct cases were found of jerk functions with four terms and one quadratic nonlinearity that have strange attractors. These are mostly generalizations of the simpler chaotic cases previously described with an additional linear term. Most if not all of them are functionally equivalent to cases in Ref. 8. For example, one can add a term proportional to $\dot{x}$ or a constant term in either Eq. (11) or (13) and find chaotic solutions. These cases are characterized by a pair of parameters, and hence it is more tedious to explore their properties.

Other examples of dissipative chaotic flows were found of the form,

$$\dddot{x}+A\ddot{x}+\dot{x}+f(x)=0, \tag{14}$$

where $f(x)$ is a second-degree (or higher) polynomial given in the notation of Eq. (8) by $f(x)=a_9x^2+a_8x+a_{10}$. One such case is

$$\dddot{x}+A\ddot{x}+\dot{x}-x^2+B=0, \tag{15}$$

which has chaotic solutions for parameters in the neighborhood of $A=0.5$ and $B=0.25$ and a period-doubling route to chaos as A is decreased or B is increased. Its attractor is shown in Fig. 3 including a portion of the trajectory as it spirals outward from an initial condition near (but not at!) one of the unstable saddle-foci at $x=-B^{1/2}$, $\dot{x}=0$, $\ddot{x}=0$ (with eigenvalues $\lambda\cong-0.8$, $0.152\pm1.105i$. Equation (15) is functionally equivalent to case S in Ref. 8.

B. Conservative systems

In contrast to the cases above, conservative systems are ones in which the phase-space volume is conserved. A sufficient but not necessary condition is $a_1=a_2=a_3=a_4=0$. The only such cases found were of this type and had the form of Eq. (14) with $A=0$. Since chaos requires a nonlinearity, the function $f(x)$, must contain a quadratic or other nonlinearity. No chaotic solutions were found for $f(x)=\pm x^2$, but with an added linear or constant term (or both), many such solutions were found. However, in each case, the trajectories eventually escaped, although the chaotic transient can persist for hundreds of cycles. Unbounded trajectories in a conservative (volume-conserving) system may seem paradoxical, but an example is a spacecraft launched from the earth with an initial velocity just sufficient for it to escape from the solar system.

An example of a conservative system that exhibits chaos is Eq. (15) with $A=0$,

$$\dddot{x}+\dot{x}-x^2+B=0. \tag{16}$$

Positive values of B less than about 0.05 produce chaotic solutions for selected initial conditions. Large values of B ($\cong$0.05) are chaotic for most initial conditions, but the trajectory quickly escapes. As B approaches zero, the range of initial conditions that produce chaos shrinks to zero, and the escape time approaches infinity. An appropriate intermediate value is $B=0.01$. As with Eq. (15) this system has two equilibrium points at $x=\pm B^{1/2}$, $\dot{x}=0$, $\ddot{x}=0$. They are both unstable saddle foci, with the trajectory spiraling out from the one at $-B^{1/2}$ and into the one at $+B^{1/2}$, producing a toroidal structure. The eigenvalues are given by the characteristic equation, $\lambda^3+\lambda\pm2B^{1/2}=0$. Equation (16) may represent the algebraically simplest example of a conservative chaotic flow, analogous to Eq. (9) for dissipative chaotic flows. It may also be the simplest formulation of a torus for suitable initial conditions.

The behavior of such a system is best exhibited in a Poincaré section, where, for example, the location of the trajectory as it punctures the $\ddot{x}=0$ plane is plotted for various initial conditions. Figure 4 shows such a plot for Eq. (16) with $B=0.01$. Twenty-one initial conditions are shown, uniform over the interval $x(0)=0$, $-0.011<\dot{x}(0)<0$, $\ddot{x}(0)=0$. The global topology is a set of nested tori produced by incommensurate periodic oscillations in x (vertical in Fig. 4)

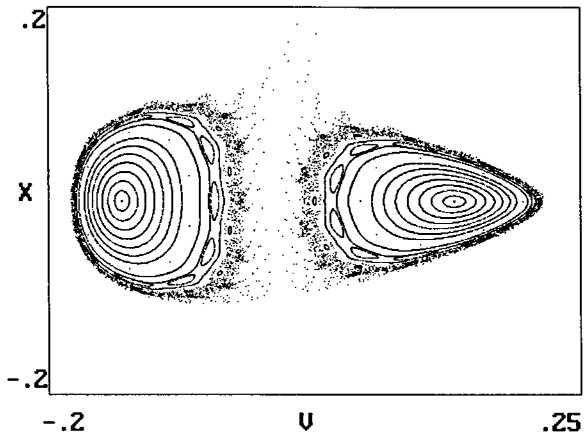

Fig. 4. Poincaré section at $\ddot{x}=0$ for Eq. (16), with $B=0.01$.

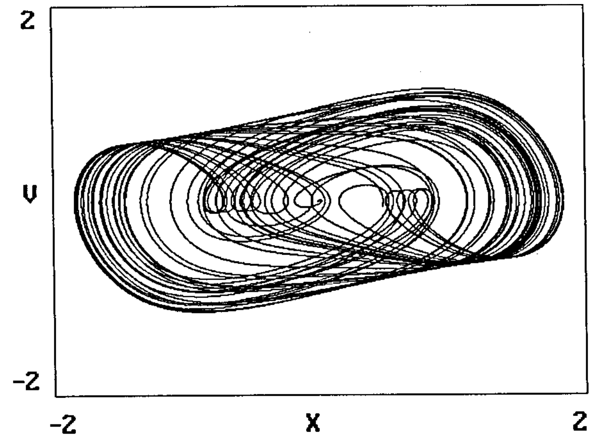

Fig. 5. Strange attractor for Eq. (21), with $A=0.25$.

and $\dot{x}$ (horizontal in Fig. 4). However, there are an infinite number of surfaces where the frequency ratio is rational, producing chains of islands in the Poincaré section. The period-8, -9, -10, and -11 islands are evident in Fig. 4. Each of these islands is surrounded by a separatrix in the vicinity of which chaos occurs. Most of these regions are invisibly small in Fig. 4. However, the period-10 and higher islands overlap, producing a large connected stochastic region that extends to infinity in the $+x$ direction. The islands with period less than 10 are apparently enclosed by KAM tori (Kolomorogov–Arnold–Moser[24]) and are thus bounded.

Equation (16) can be transformed into a form that resembles the logistic equation in Eq. (1) by defining a new variable $y \equiv x/A + 1/2$, where $A \equiv 2B^{1/2}$. The resulting equation,

$$\dddot{y} + \dot{y} + Ay(1-y) = 0, \tag{17}$$

has properties identical to Eq. (16) except for a scaling factor. A value of $A=0.2$ gives results analogous to $B=0.01$ in Fig. 4.

One usual characteristic of conservative flows, not shared by dissipative flows is time-reversal invariance. Equation (16) has this property as can be verified by replacing t with $-t$ and defining a new variable $y \equiv -x$. The resulting equation for y is identical to Eq. (16). Similarly, Eq. (17) is time-reversal invariant as can be verified by replacing t with $-t$ and defining a new variable $x \equiv 1-y$.

V. NEWTONIAN JERKS

The jerk function in Eq. (8) is the most general quadratic polynomial form, but it is not in general derivable by differentiating Newton's second law with a force that depends explicitly on the instantaneous position, velocity, and time. For such a case, we require that $F(\dot{x},x,t)$ satisfy

$$dF/dt = \ddot{x}\partial F/\partial \dot{x} + \dot{x}\partial F/\partial x + \partial F/\partial t = mj, \tag{18}$$

which in turn implies that $\dot{F}$ (and hence j) must be of the form $\dot{F} = \ddot{x}U + \dot{x}V + c$, where $U \equiv \partial F/\partial \dot{x}$ and $V \equiv \partial F/\partial x$. If j is of the form $j = j(x,\dot{x},\ddot{x})$, then c is a constant. If x and $\dot{x}$ are independent, then $\partial U/\partial x = \partial V/\partial \dot{x}$, and Eq. (8) reduces to

$$j = (a_1 + a_2 x + a_3 \dot{x})\ddot{x} + (a_5 + a_6 x + a_2 \dot{x})\dot{x} + a_{10}. \tag{19}$$

Equation (19) is the most general quadratic form of what might be called a Newtonian jerk function.

An extensive search for chaotic solutions of quadratic Newtonian jerk equations did not produce any such examples, even with all six coefficients not zero. This result is reasonable since the force must have an explicit time dependence (unless $a_{10}=0$), and this dependence consists of an additive term of the form ct, which is unbounded as t approaches infinity. However, the total force must be bounded since it is proportional to $\ddot{x}$, which is bounded.

VI. CUBIC NONLINEARITIES

The procedure outlined above can be extended to other types of nonlinearities. For example, consider jerk functions with only cubic nonlinearities. To impose symmetry, set the constant and quadratic (even) terms to zero. The most general such jerk function is

$$j = (b_1 + b_2 x^2 + b_3 \dot{x}^2 + b_4 \ddot{x}^2 + b_5 x\dot{x} + b_6 x\ddot{x} + b_7 \dot{x}\ddot{x})\ddot{x} + (b_8 + b_9 x^2 + b_{10}\dot{x}^2 + b_{11} x\dot{x})\dot{x} + (b_{12} + b_{13} x^2)x. \tag{20}$$

Note that Eq. (20) is not a Newtonian jerk because it contains terms higher than linear in $\ddot{x}$.

A search was carried out for chaotic solutions using this jerk form. The search was less extensive than the one with quadratic jerks (about 10^6 cases vs 10^7). However, chaotic solutions were found about ten times more often (about 0.1% of the cases examined vs 0.01%), and so many examples were found. Eight functionally distinct forms were found with three terms and two cubic nonlinearities, and four were found with four terms and one cubic nonlinearity. Interestingly, no cases as simple as Eq. (9) or (16) were found, although it's difficult to rule out their existence.

An example of a cubic dissipative chaotic flow that occurred often and that has a different structure than the cases previously described is

$$\dddot{x} + \ddot{x}^3 + x^2\dot{x} + Ax = 0. \tag{21}$$

It is governed by a single parameter A, which over the range $0<A<1$ produces limit cycles of many periodicities interspersed within broad regions of chaos. Figure 5 shows its attractor for $A=0.25$, projected onto the $x-v$ plane including a portion of the trajectory as it spirals outward from an initial condition near (but not at!) the unstable saddle focus at the origin, with eigenvalues $\lambda = (-A)^{1/3}$. Similar appearing cha-

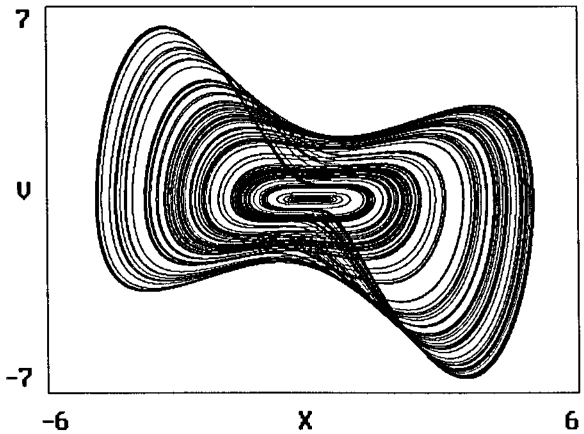

Fig. 6. Strange attractor for Eq. (22), with $A=3.6$.

otic solutions can be found by replacing the $\ddot{x}^3$ term in Eq. (21) by $x^2\ddot{x}$ or by $x\ddot{x}^2$.

Another dissipative cubic jerk function that has chaotic solutions is

$$\dddot{x}+A\ddot{x}-x\dot{x}^2+x^3=0. \tag{22}$$

Its strange attractor for $A=3.6$ as shown in Fig. 6 projected onto the $x-v$ plane resembles two back-to-back elongated Rössler attractors. The origin is a saddle focus with eigenvalues $\lambda=0, 0, -A$. It is linearly neutrally stable, but weakly unstable because of higher-order nonlinearities.

A cubic system was found that is conservative and chaotic. It has three terms and two cubic nonlinearities,

$$\dddot{x}+x^2\dot{x}-A(1-x^2)x=0. \tag{23}$$

It consists of two sets of nested tori, one at positive x and the other at negative x, coupled in such a way that trajectories near their intersection are chaotic and encircle both tori. The trajectories are bounded, in contrast to the case in Fig. 4. Its Poincaré section in the $\ddot{x}=0$ plane for $A=0.01$ is shown in Fig. 7. Twenty-one initial conditions are shown, uniform over the interval $-0.769<x(0)=0.65\dot{x}(0)<0.769$, $\ddot{x}(0)=0$. Island structure is just barely discernible near the last closed toroidal surface.

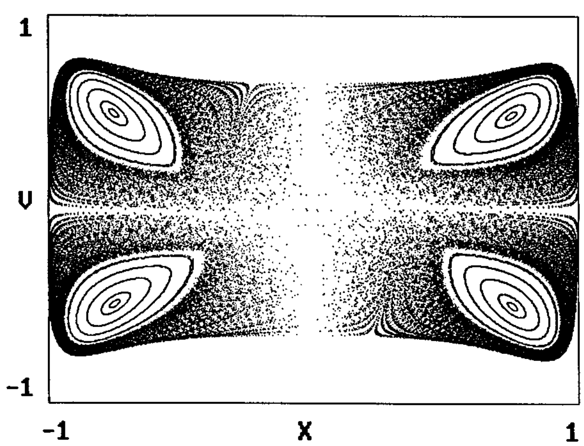

Fig. 7. Poincaré section at $\ddot{x}=0$ for Eq. (23), with $A=0.01$.

VII. SIMPLE NUMERICAL METHOD

It is worth noting that all the results in this paper can be replicated using an extremely simple numerical algorithm. The difficulty in obtaining reliable solutions of coupled differential equations has inhibited teachers and students from devoting the same attention to chaotic flows as has been given to chaotic maps such as the logistic map. Inappropriate numerical methods can produce spurious results, including false indications of chaos, and the temptation is to rely on pedagogically undesirable canned algorithms.

Consider a linear harmonic oscillator, $\ddot{x}=-x$, whose phase-space trajectory is a circle with a radius determined by the initial conditions and proportional to the square root of the energy. The most straightforward way to solve such an equation is the Euler method,

$$x_{n+1}=x_n+hv_n, \quad v_{n+1}=v_n-hx_n, \tag{24}$$

where h is a small increment of time. However, it is easy to show that each iteration causes the radius to increase by a factor of $1+h^2/2$ and the energy by a factor of $1+h^2$. Since $2\pi/h$ iterations are required to complete one cycle, the cumulative error is linear in h, and hence the method is called first order. The trajectory spirals outward to infinity for any choice of h.

This problem is not as serious as it appears for dissipative systems since it merely reduces the dissipation by an amount that can be made negligible by choosing h sufficiently small. However, for a conservative system, the Euler method is essentially useless if the trajectory is followed for many cycles. A small change in Eq. (24) suggested by Cromer[25] leads to a system that conserves energy exactly when averaged over half a cycle:

$$x_{n+1}=x_n+hv_n, \quad v_{n+1}=v_n-hx_{n+1}. \tag{25}$$

This form also is very easy to program because it allows the variables to be advanced sequentially rather than simultaneously. It generalizes to higher dimensions and performs well with jerk systems since two of the derivatives involve only a single variable. A simple (DOS) BASIC program that solves Eq. (9) by this method is

```
SCREEN 12
x=.02
v=0
a=0
h=.01
WHILE INKEY$='' ''
  x=x+h*v
  v=v+h*a
  j=-2.017*a+v*v-x
  a=a+h*j
  PSET (320+40*x, 240-40*v)
WEND
```

This program produces the attractor in Fig. 1. Although all the chaotic systems described in this paper were verified with a fourth-order Runge–Kutta integrator, the figures were produced with minor variations of the code above (sometimes with a much smaller value of h) to emphasize the usefulness of this simple algorithm and to encourage experimentation. For long calculations, double-precision (or higher) is recommended to control round-off errors.

VIII. SUMMARY

This paper has shown many examples of previously unknown chaotic systems that involve a third-order ODE in a single variable with simple polynomial (quadratic and cubic) nonlinearities and either one or two control parameters. None of the cases have been examined in great detail, offering the opportunity for additional exploration. One could search for other similar and perhaps even simpler examples of chaotic flows. One could look at other nonlinearities, such as trigonometric, logarithmic, or exponential. The bifurcations and routes to chaos could be examined. The basins of attraction could be mapped. The Lyapunov exponents and dimensions could be calculated. The structure of various Poincaré sections and return maps could be studied. One could try to construct physical models to which these equations apply and attempt to observe their chaotic behavior. These suggestions represent a wealth of possibilities for student research projects. The simple computer code described in the previous section provides an appealing starting point for such studies.

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FACULTY SALARIES

There is also too small a compensation allowed to the Professors to make it an object for men of talents to settle down in this business. The professor in a college is obliged to see much company—the parents and guardians of the students expect some attention when they visit the place. Also the price of all articles of living in the vicinity of a college is greater than that in the country around while the salary is generally so small that with the strictest economy the ends cannot be made to meet at the close of the year. We have too many colleges. The endowments are too much scattered to produce the best effect or to allow of salaries which shall secure competent instructors and the necessary implements of education. The salaries at Yale are but 12 hundred dollars and those at Schenectady were the same until lately they have been cut down. In our college none of the Professors are able to live on their salaries. Such is the expense of living in this place that since I have been in Princeton I have been obliged to expend from 250 to 300 dollars per year more than I receive from the college. The trustees however are desposed to be as liberal as the state of the funds will allow but they cannot exceed their means.

Joseph Henry, letter to Peter Bullions (1846), in *The Papers of Joseph Henry*, edited by Marc Rothenberg (Smithsonian Institution Press, Washington, 1992), Vol. 6, pp. 461–462.