A unit on oscillations, determinism and chaos for introductory physics students

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This article describes a unit on oscillations, determinism and chaos developed for calculus-based introductory physics students as part of the laboratory-centered Workshop Physics curriculum. Students begin by observing the motion of a simple pendulum with a paper clip bob with and without magnets in its vicinity. This observation provides an introduction to the contrasting concepts of Laplacian determinism and chaos. The rest of the unit involves a step-by-step study of a pendulum system that becomes increasingly complex until it is driven into chaotic motion. The time series graphs and phase plots of various configurations of the pendulum are created using a computer data acquisition system with a rotary motion sensor. These experimental results are compared to iterative spreadsheet models developed by students based on the nature of the torques the system experiences. The suitability of the unit for introductory physics students in traditional laboratory settings is discussed.
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I. INTRODUCTION

Many contemporary fields of physics require a knowledge of quantum mechanics or relativity. For this reason most calculus-based introductory physics courses rarely give students any real insight into emerging fields of research. The fact that the field of nonlinear dynamics is almost entirely classical in nature provides us with an opportunity to give students first-hand experience with an active field of contemporary physics research. For this reason, we have developed a unit on *Oscillations, Determinism and Chaos*[1] as a culminating experience for calculus-based introductory physics students as they complete the mechanics portion of the Workshop Physics curriculum.[2,3]

A. The Workshop Physics Project

The Workshop Physics Project began in the fall of 1986 with a grant from the Fund for Improvement of Postsecondary Education (FIPSE). As a result of continued support from both FIPSE and the National Science Foundation, curricular materials have been produced including an Activity Guide,[4] computer hardware and software, and apparatus to help instructors teach introductory physics without lectures. The major objective of Workshop Physics courses is to help students understand the basis of knowledge in physics as a subtle interplay between observations, experiments, definitions, mathematical descriptions, and the construction of theories. To this end, students use the Activity Guide to make predictions and observations, do guided derivations, and learn to use flexible computer tools to develop mathematical models of phenomena.

Instead of spending time in lectures and separate laboratory sessions, students in calculus-based Workshop Physics courses center their work on the Activity Guide. The four modules of the Guide contain 28 units covering topics in mechanics, thermodynamics, electricity and magnetism, and nuclear physics. At Dickinson College students spend 6 hours a week in a laboratory environment, and are able to complete 27 of these units in two semesters—approximately 1 unit each week. Although Workshop Physics students spend an equivalent amount of time solving problems and doing equation verification experiments as those who study under the lecture method, they have considerably more experience making observations, collecting data, and using computer tools.

B. The role of the Oscillations, Determinism and Chaos unit

The unit on *Oscillations, Determinism and Chaos*[5] completes a series of 15 mechanics units that cover kinematics, Newton's laws, momentum, mechanical energy, rotational motion, and simple harmonic motion. Most of the laboratory work in the final unit on chaos involves recording and analyzing the motion of a physical pendulum that is made increasingly complex until it becomes chaotic.

In previous units, students gain considerable experience with mathematical modeling by using the dynamic graphing capability of Excel® to fit their data to analytic functions (linear, quadratic, inverse, and sinusoidal). The chaos unit introduces students to the use of the spreadsheet to model more complex systems using the Euler method for numerical integration. Students also use (but do not develop) a spreadsheet-based second-order Runge–Kutta method to explore other possible behaviors of their chaotic pendulum system and to test the sensitivity of the system to initial conditions.

Because an overarching goal of the chaos unit is to explore the viability of Laplacian determinism, the unit serves both a philosophical and theoretical capstone to the study of Newtonian mechanics.

II. THE CHAOTIC PHYSICAL PENDULUM SYSTEM

A. The experimental apparatus

The apparatus that students spend most of their time using is a physical pendulum consisting of an aluminum disk mounted on the low friction shaft of a rotary motion sensor. This sensor is a digital encoder that transmits up to 1440 logic pulses per revolution to a digital interface. When a small mass is bolted to the edge of the disk and displaced from vertical equilibrium, the system becomes a physical

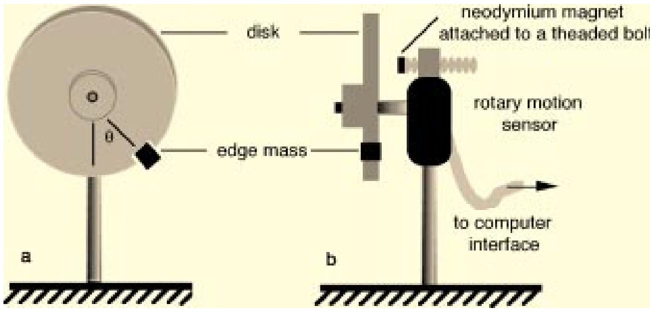

Fig. 1. The basic physical pendulum with adjustable eddy damping.

pendulum [Fig. 1(a)]. Adjustable eddy damping is added by means of a small magnet attached to a threaded bolt [Fig. 1(b)].

Students can modify the pendulum so that a string, springs, and a driver motor are coupled to it via a small drive wheel attached to the pendulum disk (Fig. 2). For certain combinations of the springs, disk mass, edge mass, eddy damping, and driver motor frequency, the pendulum becomes chaotic.

B. Commercially available chaotic pendula

In 1989 Priscilla Laws, Desmond Penny, and Brock Miller began developing the chaotic physical pendulum system at Dickinson College. We used a rotary encoder developed by Robert Teese and Ronald Thornton and a data acquisition system distributed by Vernier Software and Technology.[6]

After several years of testing in Workshop Physics courses, personnel at PASCO improved and adapted components of the Dickinson College apparatus for use with their own driver motor and data acquisition system. These components are available for the study of large angle oscillations, magnetic damping, driven harmonic motion, and chaotic motion.[7,8] The PASCO pendulum apparatus, when used with a relatively low cost data acquisition system (distributed by either PASCO or Vernier Software and Technology), is suitable for use with our chaos unit.

There are at least two other chaotic dynamical systems that can be purchased, including the Klinger Torsion Pendulum[9] and the Daedalon Chaotic pendulum.[8,10] However, the use of either of these systems in an introductory physics laboratory would require a significant modification of our curricular materials.

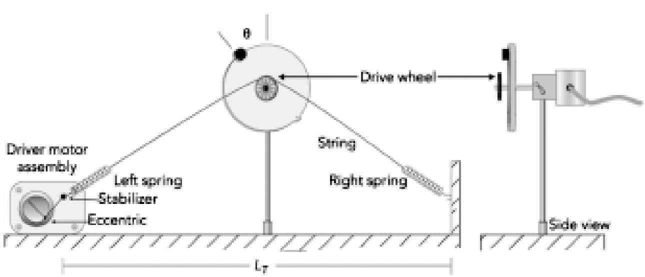

Fig. 2. A string attached to springs and a driver motor is wrapped around a drive wheel consisting of a smaller plastic disk attached to the physical pendulum disk.

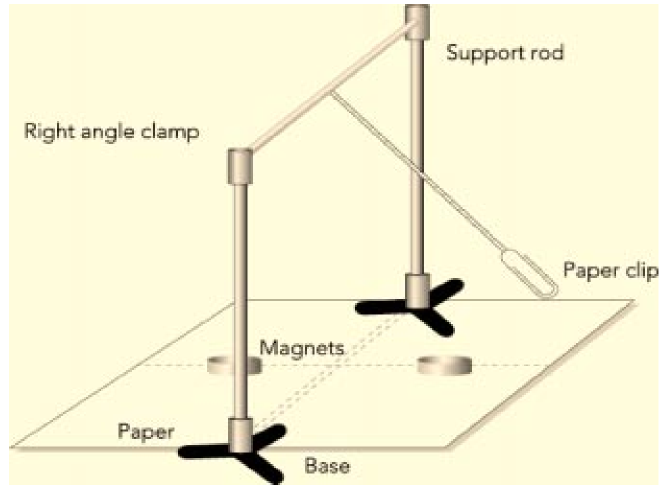

Fig. 3. This paper clip pendulum bob is reasonably insensitive to initial conditions if no magnets are present.

III. THE INTRODUCTORY CHAOS UNIT

The unit is designed to fit within the 2-hour sessions that are typically used in Workshop Physics courses. Although it requires about 8 hours of student time to complete, the first 2 hours of activities do not require access to a laboratory and can be done independently.

Session One: An Introduction to Chaos. Students are asked to read several pages of introductory material in which the concept of a dynamical system is introduced. The following quote by Pierre LaPlace is presented. "If an intellect were to know ... all the forces that animate nature and the conditions of all the objects that compose her, and were capable of subjecting these data to analysis, then this intellect would encompass in a single formula the motions of the largest bodies in the universe as well as those of the smallest atom; and the future as well as the past would be present before its eyes."[11]

Because students have just completed a study of simple dynamical systems for which the forces between objects in the system are well understood, their first activity is to write a short essay about the viability of using Newton's laws to predict the state of the universe assuming that the forces of interaction between all the objects in the universe are known.

Next students are asked to imagine whether or not the motion of the falling leaf in a closed box acting in the presence of known forces would be predictable in light of the following quote by Henri Poincaré: "It may happen that small differences in the initial conditions produce very great ones in the final phenomena. A small error in the former will produce an enormous error in the latter. Prediction becomes impossible... ."[12]

The 1-hour videotape produced by NOVA in 1989 entitled "*The Strange New Science of Chaos*" shows many examples of chaotic systems in different fields of study and provides an overview of the emerging techniques for studying chaotic systems.[13] Students view this video and answer some basic question about it.

The session ends with students observing the sensitivity of the subsequent motions of a paper clip pendulum with and without magnets present (Fig. 3).

Session Two: Large and Small Angle Pendulum Oscillations. In this session and the two that follow, students build

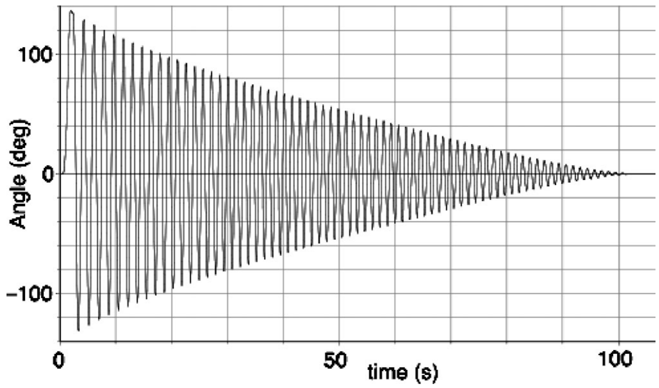

Fig. 4. The physical pendulum is displaced by an angle of 135° and released. The oscillations take just under 2 minutes to die out.

and explore the deterministic and chaotic behavior of the physical pendulum system shown in Figs. 1 and 2. They also keep track of the forces acting on the system and use these forces to develop either analytical or iterative models.

The session begins with a series of qualitative predictions and observations of the motion of the disk mounted on a low friction bearing first without and then with various edge masses. Then students collect data using the rotary motion sensor and a computer data acquisition system. They also create time series plots of the angular displacement of the pendulum as well as phase plots of rotational velocity as a function of the angular displacement. When students release an edge mass from an angle of about 135°, the resulting oscillations take about two full minutes to die out as shown in Fig. 4.

Students predict how the motions of two different runs of data will compare if they carefully start the pendulum in exactly the same way. They are not surprised to find that with some practice graphs of two identical runs match each other almost perfectly. At this point, students also are introduced to reconfiguring the data acquisition software to produce phase plots (rotational velocity versus angular position) of their matched data sets. The data acquisition software uses a smoothed first derivative of the angle versus time data to create the rotational velocity versus time data. The phase plot for one data set is shown in Fig. 5.

Session Two ends with a series of activities designed to help students compare the characteristics of large and small

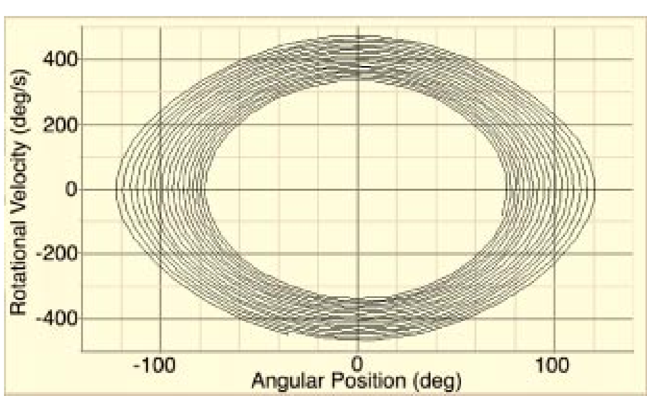

Fig. 5. A phase plot for the physical pendulum for about 25 s of oscillations. Note that the plot is nonelliptical at first and then becomes more elliptical at smaller angles.

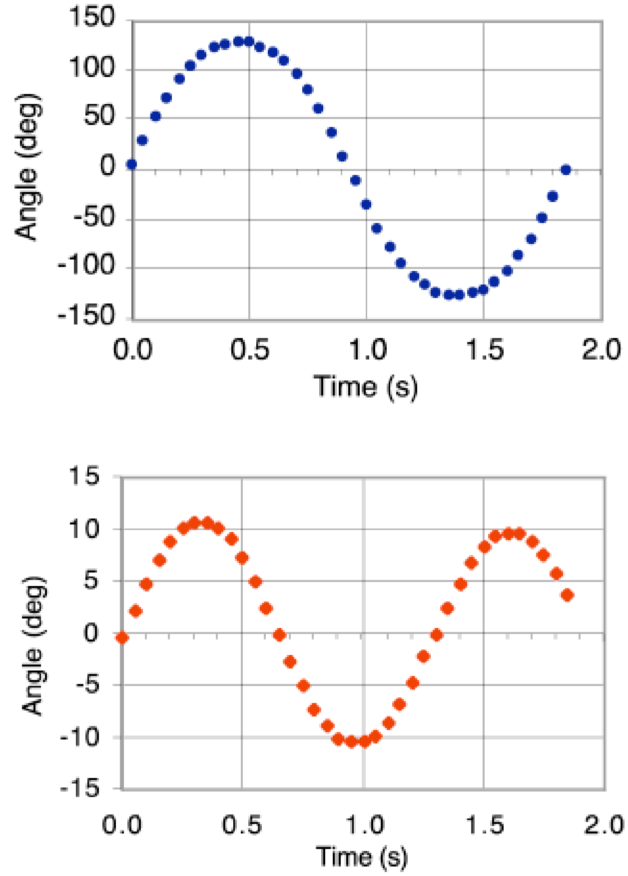

Fig. 6. A large angle cycle of period 1.8 s from the Fig. 4 data (top graph) compared to small angle oscillations that occur in the same time (bottom graph). The small angle oscillation is sinusoidal and has a shorter period.

angle motions of their physical pendulum. At first, students are asked to predict how the period of the physical pendulum and the shape of a single cycle of position versus time might differ at small and large amplitudes. To save time, portions of the data graphed in Fig. 4 are re-plotted by an instructor to enable students to compare the periods and describe how the shape of the time series graph for a single pendulum oscillation at large amplitude differs from the graph of a small amplitude oscillation (see Fig. 6). Students often are surprised that the period is longer at large amplitudes than at small amplitudes and that the large amplitude angle versus time shape is not sinusoidal. Students note that the peaks are broader at the large angular displacements than the sinusoidal plot of the angular displacement at small angular displacements.

Session Three: Using Iterations to Model the Motion. This session begins by preparing students to model their own large angle physical pendulum data, which are similar to the data shown in Fig. 3. Toward the end of the session, students add eddy damping to their pendulum system and then collect additional data. At the end of the session they are able to model their new data for the damped system by modifying the force term in their spreadsheet model (described below) to take the velocity dependent eddy damping force into account. For various reasons this session turns out to be the toughest one in the entire unit.

Students are first asked to review the derivation of the differential equation that describes small angle simple pendulum motion. As an extension to their derivation in the

Constants:	Edge Mass	m =	1.00E-02	[kg]
	Disk Mass	M =	0.143	[kg]
	Disk Radius	R =	5.00E-02	[m]
	Rotational Inertia	I =	2.04E-04	[kg•m^2]
Time Step:	Iteration Interval	Δt =	0.050	[s]
Initial Conditions:	Angular Position	θ_o=	2.250	[rad]
	Rotational Velocity	ω_o=	0.600	[rad/s]

	DATA	ITERATIVE EQUATIONS		
	[rad]	[rad]	[rad/s]	[rad/s^2]
Time (s)	**θ-data**	**θ-model**	**ω-model**	**α-model**
0.00	2.25	2.25	0.60	-18.72
0.05	2.23	2.23	-0.34	-18.97
0.10	2.17	2.17	-1.28	-19.88
0.15	2.06	2.06	-2.28	-21.29
0.20	1.89	1.89	-3.34	-22.86
0.25	1.67	1.66	-4.49	-23.96
0.30	1.39	1.38	-5.68	-23.62
0.35	1.06	1.04	-6.86	-20.70
0.40	0.67	0.64	-7.90	-14.39
0.45	0.24	0.21	-8.62	-5.02
0.50	-0.19	-0.23	-8.87	5.56
0.55	-0.62	-0.66	-8.59	14.81
0.60	-1.01	-1.06	-7.85	20.94
0.65	-1.35	-1.40	-6.81	23.69
0.70	-1.63	-1.68	-5.62	23.92

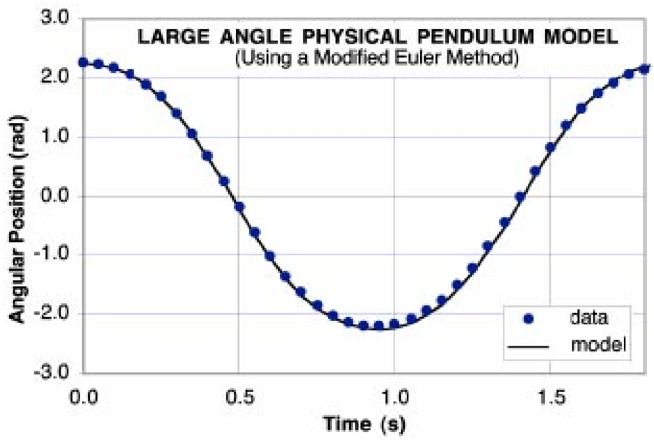

Fig. 7. A spreadsheet showing an overlay graph of data points and a curve representing the theoretical relation between angular position and time for one cycle of a physical pendulum oscillating with an amplitude of about 135°.

previous unit on harmonic motion, we give students some hints that enable them to determine the differential equation that describes the motion of the physical pendulum oscillating with a large amplitude. They find that

$$\alpha(t)=\frac{\tau^{\mathrm{net}}}{I}=\frac{\tau^{\mathrm{grav}}}{I}=-\left(\frac{mgR\sin(\theta(t))}{I}\right), \tag{1}$$

where

$$I=mR^2+\tfrac{1}{2}MR^2 \tag{2}$$

is the rotational inertia of a disk of mass M and radius R that has an edge mass of mass m located a distance R from the center of the disk.

Because the differential equation for large angle motion cannot be easily solved analytically, we introduce students to a modified Euler method[14]—an iterative numerical integration scheme for using the equation of motion to predict the rotational acceleration, velocity, and position of the pendulum as a function of time.[15]

Our modified Euler method involves a step through time that starts with the initial values for the pendulum's angular displacement and rotational velocity. This iterative method involves the use of Eq. (1) and two additional equations derived from the definitions of rotational acceleration and velocity. The first additional equation is based on the definition of rotational acceleration ($\alpha(t)\equiv d\omega/dt\approx\Delta\omega/\Delta t$) and is given by

$$\omega(t+\Delta t)\approx\omega(t)+\alpha(t)\Delta t. \tag{3}$$

The second equation is based on the definition of rotational velocity ($\omega(t)\equiv d\theta/dt\approx\Delta\theta/\Delta t$) and is given by

$$\theta(t+\Delta t)\approx\theta(t)+\omega(t+\Delta t)\Delta t. \tag{4}$$

Equations (3) and (4) are good approximations to the original differential equations for small enough Δt.

To perform the iterative calculations, students begin by substituting the initial value of the angular position $\theta(0)$ into Eq. (1) to determine the initial rotational acceleration $\alpha(0)$. Next a small time interval Δt (such as 1/20th of a second) is chosen and used in Eq. (3) along with the calculated value of $\alpha(0)$ and the initial value of rotational velocity $\omega(0)$ to find a new value of the rotational velocity $\omega(0+\Delta t)$ at a time Δt later. Then the new value $\omega(0+\Delta t)$ is used in Eq. (4) to find a new value of the angular displacement $\theta(0+\Delta t)$ at time Δt. This process is repeated many times to find updated values of the rotational acceleration, rotational velocity, and angular displacement. Once the spreadsheet is set up properly, the computer does all the iterative calculations and graphing.

To minimize the errors associated with the Euler method, we ask students to model their angle versus time data at a time when the pendulum's angular displacement is a maximum so that the initial rotational velocity is close to zero. Before starting the modeling, students are advised that they must transform their angular displacement data from degrees to radians.

Students use a spreadsheet template that we provide them (along with much instructor and teaching assistant advice) to create a model to their data like the one shown in Fig. 7.[16]

Adding magnetic damping forces to the physical pendulum system: In the next part of the session, students position a damping magnet very close to the face of the aluminum disk to create significant eddy damping and a real time graph of angular position versus time for an initial angular position of about 135°. Then students were shown how to add the term $\tau^{\mathrm{damp}}=-b\omega$ to Eq. (1) (the torque equation used in the iterative calculations). Adding this term and copying it down through the column in which it appears gives students instant

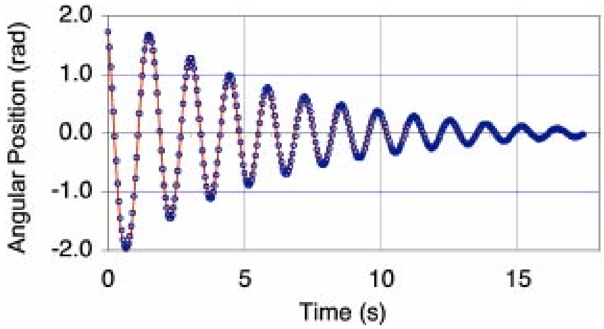

Fig. 8. An overlay graph of data points and a curve representing the theoretical relationship between angular position and time for the physical pendulum oscillating in the presence of eddy damping. Data are shown by open circles, the line represents the modified Euler model of the data.

results for their new model. By using the damping coefficient b as an adjustable parameter, students can obtain an excellent fit to their data like that shown in Fig. 8.

Session Four: The Chaotic Physical Pendulum. In this final session students modify their pendulum so that a string, springs, and a driver motor are coupled to the disk and edge mass as depicted in Fig. 2.

Exploring the natural frequencies of the system: Students are asked to observe the natural oscillation frequencies of the apparatus when it is configured in different ways. These observations help them understand why the system motion becomes chaotic when it is driven at certain frequencies. Students start by observing and determining the frequency of oscillation of the disk without the edge mass added as it moves under the influence of torques caused by springs wrapped around the drive wheel (Fig. 2). Next they configure the system as a pendulum by adding a small edge mass to it and measure the natural frequency of the pendulum without the springs. Then students re-attach the springs to the driver wheel of the pendulum and re-balance the system so the springs are stretched equally when the mass is perched straight up on the top of the disk at its unstable equilibrium point (Fig. 9).

Next, students measure the left and right equilibrium angles θ_L and θ_R with respect to a vertical axis as shown in Fig. 2. If the springs are properly balanced, the magnitudes of these two angles are essentially the same. Finally students measure the natural frequency of oscillation of the spring–pendulum system when the edge mass has fallen to the right of its highest possible position and again when it has fallen to the left.

Driving the system at natural frequencies: Students are asked to set the drive frequency of their electric motor to one of the natural frequencies they have measured, balance the springs so the edge mass points straight up, turn on the motor, and collect data for the angular displacement versus time. Students find that whenever the motor is near a natural frequency, the system settles rather quickly into a stable oscillation mode.

Driving the system chaotic: In the next activity students set the drive frequencies so that they are different from any of the natural frequencies and see if they can achieve a situation in which there is an irregular pattern in the time series graph depicting the angular position versus time. A typical pattern is shown in Fig. 4 for the time series graph and the phase plot of the chaotic physical pendulum system.

Students find that their systems are so sensitive to the initial values of the angular position and rotational velocity that it is impossible for them to recreate the initial conditions accurately enough to repeat a pattern on either a time series graph or a phase plot for more than a few seconds. A typical example of this sensitivity is shown in Fig. 10.

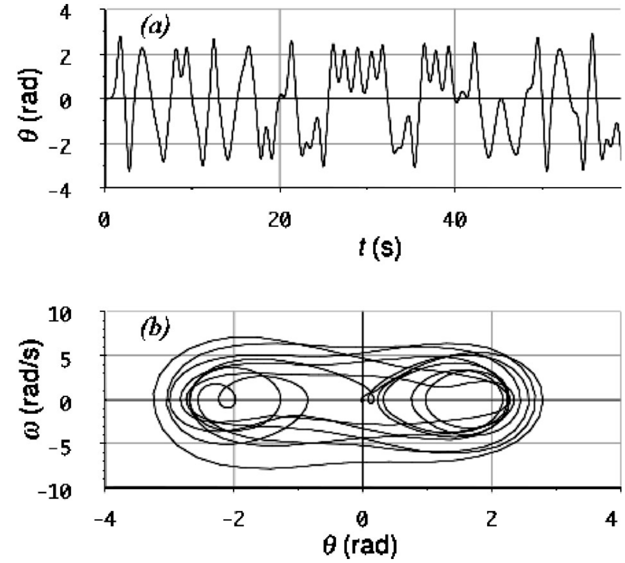

Fig. 9. (a) Rotary motion sensor data for the angular displacement of a chaotic physical pendulum vs time is shown in the top graph; (b) phase plot depicting rotational velocity vs angular position for the first 30 s of motion of the same pendulum.

Using an iterative model of the chaotic pendulum motions: Students are led through a guided derivation of the four torques that act on the disk of the pendulum, including the gravitational torque on the edge mass, the eddy damping torque exerted on the aluminum disk by the magnet, the spring torques, and the torque exerted by the driver. We write the net torque as

$$\tau^{\text{net}} = \tau^{\text{grav}} + \tau^{\text{damping}} + \tau^{\text{spring}} + \tau^{\text{driver}}. \tag{5}$$

The rotational acceleration is given by the net torque divided by the rotational inertia of the physical pendulum, or

$$\alpha = \frac{\tau^{\text{net}}}{I}, \tag{6}$$

where the rotational inertia of the pendulum is given by Eq. (2). The notation used for the quantities needed in the model are summarized in Table I. It can be shown that the torques are given by

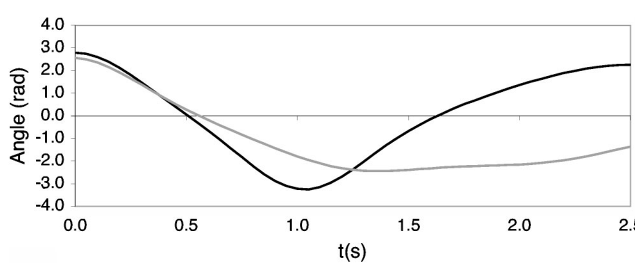

Fig. 10. An overlay time series graph of the first few seconds of two different runs of chaotic physical pendulum. Both sets of data are recorded for similar initial conditions. Note that the two motions begin to diverge from each other within seconds.

Table I. Summary of notation.

Symbol	Name	Typical value
m	Edge mass	0.010 kg
R	Disk radius	0.050 m
M	Disk mass	0.143 kg
g	Gravitational constant	9.8 m/s^2
θ	Angular displacement of the edge mass from upward vertical with positive left displacement	Variable (rad)
ω	Rotational velocity of the edge mass	Variable (rad/s)
b	Magnetic damping coefficient	6.0×10^{-5} ((ms)/rad)
r	Axle radius	0.025 (m)
A_d	Driver amplitude	0.032 (m)
T_d	Driver period	1.56 (s)
t	Current time	Variable (s)
ϕ	Phase of the driver (assumed to be zero in the model)	0.0 (rad)

$$\tau^{\mathrm{grav}} = mRg \sin\theta, \tag{7}$$

$$\tau^{\mathrm{damping}} = -b\omega, \tag{8}$$

$$\tau^{\mathrm{spring}} = -2kr^2\theta, \tag{9}$$

$$\tau^{\mathrm{driver}} = +krA_d \cos[(2\pi/T_d)t + \phi]. \tag{10}$$

In principle, students can now develop an iterative spreadsheet model to describe the motion of the chaotic pendulum system. However, developing this model requires many hours of careful work which is not very instructive. In addition, the pendulum often obtains high rotational velocities as it whips back and forth. This motion means that the Euler method students had used for numerical integration will accumulate integration errors unless the time steps are extremely small. For this reason the author used the second-order Runge–Kutta integration.[17]

Students use the Runge–Kutta spreadsheet[16] to explore the theoretical behavior of their pendulum. In particular, they are asked to run the model and devise a method for describing the sensitivity of its output to small changes in the initial conditions (that is, the angular displacement and the rotational velocity at time $t=0$). A sample screen shot of the output is shown in Fig. 11. Students observe that the time series graphs and phase plots are similar to those that they found. They also find that the motion of the theoretical system also is very sensitive to the initial values of angle and rotational velocity.

Revisiting of the concept of determinism. After finishing their work with their simulations of chaotic motion, the students are asked to read and consider the meaning of a short statement that summarizes the conditions for chaotic motion: (a) It takes three or more independent dynamical variables to describe the state of the system at any given time, and (b) the equation describing the net force or torque on the system must have nonlinear term that couples several of the variables.[18] These two statements do not require students to revise their concept of determinism.

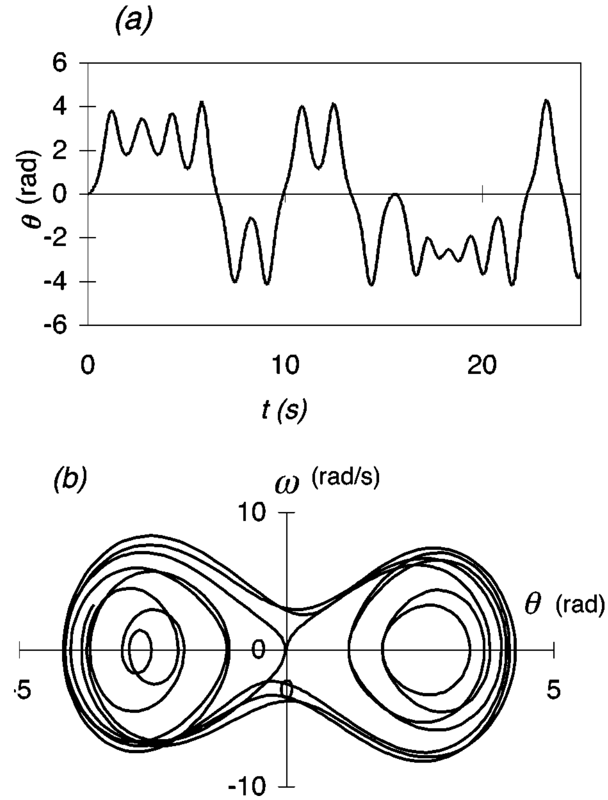

Fig. 11. Samples of (a) time series and (b) phase plot graphs generated by a second-order Runge–Kutta solution of the possible motion of a chaotic physical pendulum. The constants are similar to those used to obtain the Fig. 10 data.

Finally, students are asked again to comment briefly on the viability of Laplacian determinism. In general, the student comments on determinism both before and after they work on the unit are disappointing. We expected students to be surprised that the state of a chaotic system is unpredictable even if the torques acting on it are known. We hoped that they could speculate about what would happen in a nonquantum world if they could measure the initial state of the system to infinite precision. Instead, students often commented that Laplacian determinism is not feasible because of quantum effects.

The question posed at the beginning and and revisited at the end of the unit needs to be worded more carefully. For example, students might be asked initially: Suppose that you could know the mass, shape, position and velocity of every object in the universe to eight significant figures, how the forces and torques between them depend on these four quantities, and that the universe is governed only by Newton's laws of motion. How well could you predict the future?'' The final question might be changed to the following: Based on what you have learned by using Newton's laws of motion and the known torques to model and predict the motion of your chaotic pendulum, what changes, if any, would you make to your answer to the first question?[19]

IV. CONCLUSIONS

Many of the topics that students need to understand and to explore the behavior of the pendulum are covered in previous units. The required measurements are similar to those used in many Workshop Physics activities on mechanics.

Certain aspects of the sample activities used from the Chaos Unit are typical of Workshop Physics sessions in that they demonstrate the interplay between predictions, observations, experiments, and analysis, using both computer data acquisition software and spreadsheet tools.

In spite of the overlap in the approach taken in the Chaos Unit with others that preceded it, the relative complexity of the pendulum system and the introduction of the iterative spreadsheet modeling are still a stretch for most students. Nevertheless, we found that the Chaos Unit is both vexing and exciting to our students. Overall, we believe that our attempt to expose introductory physics students to profound aspects of contemporary physics is well worth the effort.

Adapting this unit to the laboratory portion of more traditional physics courses would require some modification. But the physics concepts that students need to understand the behavior on the chaotic pendulum are covered in the lecture portion of many calculus-based introductory physics courses. Students would need to have prior experience in earlier laboratory sessions with computer data acquisition software and be exposed to the process of fitting their data to analytical functions using spreadsheets or other software tools. In this case, this unit could be adapted for use in the last three or four laboratory periods at the end of a mechanics laboratory sequence.

In this introductory treatment of chaotic dynamics, we do not attempt to find the Lyapunov exponents needed to verify that the pendulum motions are truly chaotic. In addition, we do not introduce students to the concept of the Poincaré section. However, if a more sophisticated data acquisition system is used, these topics can be introduced in an advanced laboratory course. For example, Robert DeSerio has developed and improved the PASCO Chaotic Physical Pendulum and has reported on the results of a rigorous experimental investigation of his system including three-dimensional phase space data, the acquisition of Poincaré sections for almost all drive phases, and the calculation of Lyapunov exponents for several chaotic system configurations.[20]

ACKNOWLEDGMENTS

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[5]Reference 4, Module 2, Unit 15.

[6]Vernier Software and Technology, 13979 SW Millikan Way, Beaverton, OR 97005-2886 and ⟨www.vernier.com⟩.

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[14]H. Gould and J. Tobochnik, *Computer Simulation Methods* (Addison–Wesley, Reading, MA, 1996), 2nd ed., pp. 41–42.

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[19]S. Kellert, a philosopher of science, has a fascinating discussion of how chaos theory presents us with unpredictable deterministic models in his book entitled *In the Wake of Chaos* (University of Chicago Press, Chicago, 1993), Chap. 3.

[20]R. DeSerio, "Chaotic pendulum: The complete attractor," Am. J. Phys. **71** (3), 250–257 (2003).