Waves in locally periodic media

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(Received 18 January 2000; accepted 7 June 2000)

We review the theory of wave propagation in one dimension through a medium consisting of N identical ''cells.'' Surprisingly, exact closed-form results can be obtained for arbitrary N. Examples include the vibration of weighted strings, the acoustics of corrugated tubes, the optics of photonic crystals, and, of course, electron wave functions in the quantum theory of solids. As N increases, the band structure characteristic of waves in infinite periodic media emerges.
[DOI: 10.1119/1.1308266]

I. INTRODUCTION

In elementary physics one encounters two kinds of wave motion: *traveling* waves, which can have any frequency, and *standing* waves, which occur only for discrete ''allowed'' frequencies. The same dichotomy persists all the way through quantum mechanics, in the form of scattering states and bound states, respectively. But there exists a third kind of wave motion, that occurs in (infinite) periodic media, for which the frequencies fall into continuous ''bands,'' separated by forbidden ''gaps.'' In the quantum context this was first noted by Kronig and Penney in the classic paper that laid the foundation for the modern theory of solids.[1]

Band structure is practically the *signature* of solid state physics, but the same phenomenon occurs, in principle, for mechanical, acoustical, electromagnetic, and even oceanographic waves—it's just that whereas an ordinary macroscopic crystal, with (say) 10^7 atomic layers, is to all intents and purposes truly periodic (continuing forever), a weighted string or a corrugated pipe or a sequence of sandbars is likely to have only a relatively small number N of repeating elements. (We shall call such a system ''locally periodic.'') In practice, therefore, they exhibit only suggestive *precursors* of the band structure characteristic of the fully periodic system.

From a theoretical standpoint locally periodic systems are more difficult to analyze because Bloch's theorem,[2] which so dramatically simplifies the periodic problem, does not apply. It is in fact quite astonishing to learn that the finitely periodic case can be solved analytically for arbitrary N. This was first discovered in the optical case by Abelès in 1950;[3] it was rediscovered in the quantum context by Cvetič and Pičman in 1981[4] (and later by several others), but to our knowledge it has not been noticed for other physical systems.

Our purpose here is to provide a unified and accessible treatment of the theory of waves in locally periodic media and to survey its applications in various branches of physics. (It is surprising how little ''cross-talk'' there has been between different specialized literature streams, and the result has been a lot of duplicated effort.) From a pedagogical point of view the most interesting dividend is the possibility of exploring the emergence of band structure as the number of ''cells'' (N) increases.

Throughout this paper we restrict our attention to *nondissipative* waves propagating in *one dimension*. In Sec. II we develop the general theory, using nonrelativistic quantum scattering as a model. In Sec. III the method is adapted to several mechanical systems: transverse waves on weighted strings, longitudinal waves on weighted rods, acoustic waves in corrugated tubes, and water waves crossing a sequence of sandbars. In Sec. IV we consider electromagnetic waves in transmission lines and photonic crystals. In Sec. V we treat the case of relativistic quantum scattering using the one-dimensional Dirac equation. Section VI concludes with general remarks and observations.

II. GENERAL THEORY: QUANTUM MECHANICS

Perhaps the simplest context is quantum mechanical scattering in one dimension, governed by the Schrödinger equation:

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x,t)+V(x)\Psi(x,t)=i\hbar\frac{\partial}{\partial t}\Psi(x,t). \tag{1}$$

Separation of variables,

$$\Psi(x,t)=\psi(x)e^{-iEt/\hbar}, \tag{2}$$

reduces this to the time-independent Schrödinger equation,

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x)+V(x)\psi(x)=E\psi(x). \tag{3}$$

A. The transfer matrix

Consider first a localized potential V, restricted to the interval (a, b); the general solution is

$$\psi(x)=\begin{cases} Ae^{ikx}+Be^{-ikx} & \text{if } x<a \\ \psi_{ab}(x) & \text{if } a<x<b \\ Ce^{ikx}+De^{-ikx} & \text{if } b<x, \end{cases} \tag{4}$$

where $k\equiv\sqrt{2mE}/\hbar$. When the time factor [Eq. (2)] is included, $A\exp(ikx)$ and $C\exp(ikx)$ represent waves propagating to the right, while $B\exp(-ikx)$ and $D\exp(-ikx)$ represent waves propagating to the left (Fig. 1).

To complete the problem, one solves Eq. (3) for $\psi(x)$ in (a, b). Then, invoking the appropriate boundary conditions at a and b [typically, continuity of $\psi(x)$ and its derivative], one obtains two linear relations among the coefficients A, B, C, and D. These can be solved for any two amplitudes in terms of the other two, and the result can be expressed as a matrix equation. Usually one chooses to write the outgoing amplitudes (B and C) in terms of the incoming amplitudes (A and D) using the so-called ''**S** matrix'':

$$\begin{pmatrix} B \\ C \end{pmatrix}=\mathbf{S}\begin{pmatrix} A \\ D \end{pmatrix}. \tag{5}$$

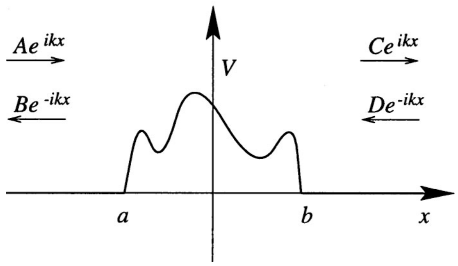

Fig. 1. Scattering from an arbitrary potential.

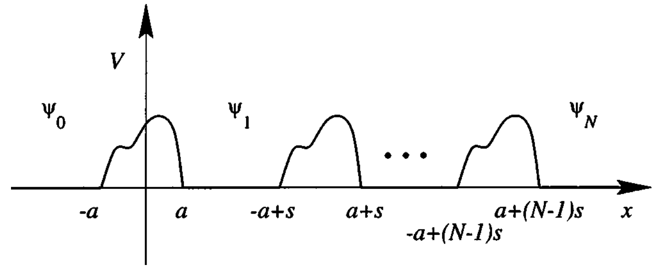

Fig. 2. A locally periodic potential.

We find it more convenient to express the amplitudes to the left of the barrier (A and B) in terms of those to the right (C and D):

$$\begin{pmatrix} A \\ B \end{pmatrix} = \mathbf{M} \begin{pmatrix} C \\ D \end{pmatrix}. \tag{6}$$

This 2×2 matrix

$$\mathbf{M} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \tag{7}$$

is called the ''transfer matrix.''[5]

Time reversal invariance and conservation of probability impose strong conditions on the structure of **M**, regardless of the specific form of the potential.[6] Taking the complex conjugate of Eq. (1) and switching the sign of t yield

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi^*(x,-t) + V^*(x)\Psi^*(x,-t)$$
$$= i\hbar \frac{\partial}{\partial t}\Psi^*(x,-t), \tag{8}$$

which is of the same form as Eq. (1) (assuming, of course, that V is real). Thus, if $\Psi(x,t)$ is a solution, then $\Psi^*(x,-t)$ [in our case $\psi^*(x)\exp(-iEt/\hbar)$] is also a solution:

$$\psi^*(x) = \begin{cases} A^* e^{-ikx} + B^* e^{ikx} & \text{if } x<a \\ \psi^*_{ab}(x) & \text{if } a<x<b \\ C^* e^{-ikx} + D^* e^{ikx} & \text{if } b<x. \end{cases} \tag{9}$$

Notice that this interchanges incoming and outgoing waves; in terms of the transfer matrix,

$$\begin{pmatrix} B^* \\ A^* \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} D^* \\ C^* \end{pmatrix}. \tag{10}$$

It follows that

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} M^*_{22} & M^*_{21} \\ M^*_{12} & M^*_{11} \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix}.$$

Comparing Eq. (6), we have

$$\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} = \begin{pmatrix} M^*_{22} & M^*_{21} \\ M^*_{12} & M^*_{11} \end{pmatrix}.$$

Evidently,

$$M_{11} = M^*_{22}, \quad M_{21} = M^*_{12}. \tag{11}$$

Next we assess the implications of conservation of probability. In one dimension the probability current,[7]

$$j \equiv \frac{\hbar}{2mi}\left(\psi^* \frac{d\psi}{dx} - \frac{d\psi^*}{dx}\psi \right),$$

is independent of x. In particular,

$$j|_{x<a} = j|_{x>b}. \tag{12}$$

Referring back to Eq. (4), this entails

$$|A|^2 - |B|^2 = |C|^2 - |D|^2. \tag{13}$$

In matrix notation,

$$(A \quad B) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} A^* \\ B^* \end{pmatrix} = (C \quad D) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} C^* \\ D^* \end{pmatrix}.$$

Using Eq. (6), we rewrite the left-hand side:

$$(C \quad D)\tilde{\mathbf{M}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{M}^* \begin{pmatrix} C^* \\ D^* \end{pmatrix}$$
$$= (C \quad D) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} C^* \\ D^* \end{pmatrix},$$

where $\tilde{\mathbf{M}}$ is the transpose of **M**. Since this is true for all C and D, it must be the case that

$$\begin{pmatrix} M_{11} & M_{21} \\ M_{12} & M_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} M^*_{11} & M^*_{12} \\ M^*_{21} & M^*_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{14}$$

This yields four constraints, but in view of Eq. (11) only one of them is really new:

$$|M_{11}|^2 - |M_{12}|^2 = 1, \tag{15}$$

or, equivalently,

$$\det \mathbf{M} = 1. \tag{16}$$

Conclusion: From the time reversal invariance of the Schrödinger equation, together with conservation of probability, it follows that all transfer matrices[8] are of the form

$$\mathbf{M} = \begin{pmatrix} w & z \\ z^* & w^* \end{pmatrix}, \tag{17}$$

where w and z satisfy

$$|w|^2 - |z|^2 = 1. \tag{18}$$

B. Multiple cells

Now let us suppose that the basic unit cell (Fig. 1) is replicated N times at regular intervals (Fig. 2). Our problem is to construct the transfer matrix for the whole array, given the transfer matrix [Eq. (17)] for a single cell.[9] Without loss of generality we assume that the unit cell is defined on the interval $(-a,a)$, and zero elsewhere. The cells are separated by a distance $s \geqslant 2a$. The wave function between the cells can be written

$$\psi_n(x) = A_n e^{ik(x-ns)} + B_n e^{-ik(x-ns)} \quad \text{for} \quad (n-1)s + a < x < ns - a, \tag{19}$$

where $0 < n < N$. By extension, $\psi_0(x) = A_0 \exp(ikx) + B_0 \exp(-ikx)$ is the wave function to the left $(x < -a)$, and $\psi_N(x) = A_N \exp[ik(x-Ns)] + B_N \exp[-ik(x-Ns)]$ is the wave function to the right $[x > (N-1)s + a]$. This notation effectively puts each $\psi_n(x)$ in a ''local coordinate system'' whose origin is the center of the next cell to the right. With the wave function written in this manner the transfer matrix for the entire array can be written as the Nth power of a ''shifted'' transfer matrix $\mathbf{P}$, constructed from $\mathbf{M}$, as follows:

For the nth cell, we have [using Eqs. (6), (17), and (19)]

$$\begin{pmatrix} A_n \\ B_n \end{pmatrix} = \begin{pmatrix} w & z \\ z^* & w^* \end{pmatrix} \begin{pmatrix} A_{n+1} e^{-iks} \\ B_{n+1} e^{iks} \end{pmatrix}, \tag{20}$$

or

$$\begin{pmatrix} A_n \\ B_n \end{pmatrix} = \mathbf{P} \begin{pmatrix} A_{n+1} \\ B_{n+1} \end{pmatrix}, \tag{21}$$

where

$$\mathbf{P} \equiv \mathbf{M} \begin{pmatrix} e^{-iks} & 0 \\ 0 & e^{iks} \end{pmatrix} = \begin{pmatrix} w e^{-iks} & z e^{iks} \\ z^* e^{-iks} & w^* e^{iks} \end{pmatrix}. \tag{22}$$

Notice that $\mathbf{P}$, like $\mathbf{M}$, is unimodular $(\det \mathbf{P} = 1)$. Using Eq. (21) recursively,

$$\begin{pmatrix} A_0 \\ B_0 \end{pmatrix} = \mathbf{P}^N \begin{pmatrix} A_N \\ B_N \end{pmatrix}, \tag{23}$$

and the whole problem reduces to the evaluation of $\mathbf{P}^N$.

There are several elegant ways to calculate the Nth power of a unimodular 2×2 matrix.[10] A particularly cute one exploits the Cayley–Hamilton theorem to establish a relation between $\mathbf{P}^2$ and $\mathbf{P}$. The characteristic equation for $\mathbf{P}$ is $\det(\mathbf{P} - \mathbf{I}p) = 0$, or

$$p^2 - p\,\mathrm{Tr}(\mathbf{P}) + \det(\mathbf{P}) = 0. \tag{24}$$

Since $\det \mathbf{P} = 1$, we can rewrite this as

$$p^2 - 2\xi p + 1 = 0,$$

where

$$\xi \equiv \tfrac{1}{2}\mathrm{Tr}(\mathbf{P}). \tag{25}$$

The Cayley–Hamilton theorem says that any matrix satisfies its own characteristic equation:[11]

$$\mathbf{P}^2 - 2\mathbf{P}\xi + \mathbf{I} = 0. \tag{26}$$

This means that any higher power of $\mathbf{P}$ can be reduced to a linear combination of $\mathbf{P}$ and the identity matrix $\mathbf{I}$:

$$\mathbf{P}^N = \mathbf{P} U_{N-1}(\xi) - \mathbf{I} U_{N-2}(\xi), \tag{27}$$

where $U_N(\xi)$ is a polynomial of degree N in ξ. Multiplying by $\mathbf{P}$,

$$\mathbf{P}^{N+1} = \mathbf{P}^2 U_{N-1}(\xi) - \mathbf{P} U_{N-2}(\xi).$$

Using Eq. (26) to substitute for $\mathbf{P}^2$:

$$\mathbf{P}^{N+1} = (2\mathbf{P}\xi - \mathbf{I}) U_{N-1}(\xi) - \mathbf{P} U_{N-2}(\xi).$$

Alternatively, putting $N \to N+1$ in Eq. (27),

$$\mathbf{P}^{N+1} = \mathbf{P} U_N(\xi) - \mathbf{I} U_{N-1}(\xi).$$

Equating these two expressions for $\mathbf{P}^{N+1}$, we obtain

$$U_{N+2}(\xi) - 2\xi U_{N+1}(\xi) + U_N(\xi) = 0. \tag{28}$$

This is the recursion relation for Chebychev polynomials. In fact, putting $N = 2$ into Eq. (27) and comparing Eq. (26), we see that $U_0(\xi) = 1$ and $U_1(\xi) = 2\xi$, so U_N is the Nth Chebychev polynomial of the second kind.[12]

Equation (27) is a closed-form expression for the Nth power of *any* unimodular 2×2 matrix. In our particular case $\mathbf{P}$ is given by Eq. (22), and

$$\begin{aligned} \xi &= \tfrac{1}{2}\mathrm{Tr}(\mathbf{P}) \\ &= \tfrac{1}{2}(w e^{-iks} + w^* e^{iks}) \\ &= \mathrm{Re}(w)\cos(ks) + \mathrm{Im}(w)\sin(ks). \end{aligned} \tag{29}$$

It follows that the transfer matrix for the whole array of N cells is

$$\mathbf{M}_N = \mathbf{P}^N \begin{pmatrix} e^{ikNs} & 0 \\ 0 & e^{-ikNs} \end{pmatrix} = \begin{pmatrix} [w e^{-iks} U_{N-1}(\xi) - U_{N-2}(\xi)] e^{ikNs} & z U_{N-1}(\xi) e^{-ik(N-1)s} \\ z^* U_{N-1}(\xi) e^{ik(N-1)s} & [w^* e^{iks} U_{N-1}(\xi) - U_{N-2}(\xi)] e^{-ikNs} \end{pmatrix}. \tag{30}$$

Conclusion: If we can determine the transfer matrix for a *single* cell (i.e., if we know w and z), we can immediately write down the transfer matrix for N cells. In particular, the transmission probability (for incidence from the left) is

$$T = \left| \frac{C}{A} \right|^2_{D=0} = \frac{1}{|M_{11}|^2}, \tag{31}$$

and hence [using Eq. (18)][13]

$$T_N = \frac{1}{1 + [|z| U_{N-1}(\xi)]^2}. \tag{32}$$

Chebychev polynomials of the second kind can be written in terms of sinusoidal functions:

$$U_N(\xi) = \frac{\sin(N+1)\gamma}{\sin \gamma}, \tag{33}$$

where

$$\gamma \equiv \cos^{-1} \xi. \tag{34}$$

In the present context, as we shall see, γ represents the ''Bloch phase'' of the corresponding (fully) periodic system.[14] The transmission coefficient [Eq. (32)] can be ex-

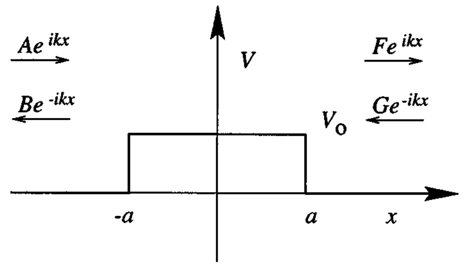

Fig. 3. Scattering from a rectangular barrier potential.

pressed in terms of the Bloch phase, avoiding explicit reference to the Chebychev polynomials:

$$T_N=\left[1+|z|^2\left(\frac{\sin N\gamma}{\sin\gamma}\right)^2\right]^{-1}. \tag{35}$$

C. Examples and applications

1. Delta functions

If the unit cell consists of a single delta function,

$$V(x)=c\,\delta(x), \tag{36}$$

then[15]

$$w=1+i\beta,\quad z=i\beta$$

where

$$\beta\equiv\frac{mc}{\hbar^2 k}. \tag{37}$$

So Eq. (29) yields

$$\xi=\cos(ks)+\beta\sin(ks), \tag{38}$$

and Eq. (32) says

$$T_N=\frac{1}{1+[\beta U_{N-1}(\xi)]^2}. \tag{39}$$

2. Rectangular barriers

Next consider the rectangular barrier (Fig. 3), with constant potential V_0 on the interval $-a<x<a$. If $E>V_0$, the wave function is:

$$\psi(x)=\begin{cases} Ae^{ikx}+Be^{-ikx} & \text{if } x<-a \\ Fe^{ik'x}+Ge^{-ik'x} & \text{if } -a<x<a \\ Ce^{ikx}+De^{-ikx} & \text{if } x>a, \end{cases} \tag{40}$$

where $k'\equiv\sqrt{2m(E-V_0)}/\hbar$. Imposing continuity of $\psi(x)$ and its derivative at $x=\pm a$ to eliminate F and G, we obtain the transfer matrix

$$\mathbf{M}=\begin{pmatrix} (\cos 2k'a-i\varepsilon_+\sin 2k'a)e^{2ika} & i\varepsilon_-\sin 2k'a \\ -i\varepsilon_-\sin 2k'a & (\cos 2k'a+i\varepsilon_+\sin 2k'a)e^{-2ika} \end{pmatrix}, \tag{41}$$

where

$$\varepsilon_\pm\equiv\frac{1}{2}\left(\eta\pm\frac{1}{\eta}\right),\quad \eta\equiv\frac{k}{k'}. \tag{42}$$

The transmission coefficient for a multicell array is given by Eq. (32), with

$$\begin{aligned} &z=i\varepsilon_-\sin(2k'a),\\ &\xi=\cos(2k'a)\cos(k\ell)-\varepsilon_+\sin(2k'a)\sin(k\ell), \end{aligned} \tag{43}$$

where $\ell\equiv s-2a$ is the distance between adjacent barriers. [For the case $E<V_0$, we simply substitute $k'\to -i\lambda$ with $\lambda\equiv\sqrt{2m(V_0-E)}/\hbar$.]

The delta function and the rectangular barrier have been studied extensively.[16] Perhaps the most striking result is the surprisingly early emergence of band-like structure, which is clearly visible with N as low as 5 (Fig. 4): In some energy ranges the transmission is close to perfect, but in the intervening gaps the wave is mostly reflected. The location of these gaps, which are especially pronounced as N increases, is predicted by the structure of ξ, the cosine of the Bloch phase γ, as indicated in the last plot of Fig. 4.

3. Bound states

If the potential in the unit cell runs *negative*, there may be discrete bound states (with $E<0$) in addition to the scattering states ($E>0$). In this case,

$$\kappa\equiv -ik=\sqrt{2m|E|}/\hbar \tag{44}$$

is real (and positive), and the wave function [Eq. (4)] takes the form

$$\psi(x)=\begin{cases} Ae^{-\kappa x}+Be^{\kappa x} & \text{if } x<a \\ Ce^{-\kappa x}+De^{\kappa x} & \text{if } x>b. \end{cases} \tag{45}$$

For a bound state we must have $A=0$ (otherwise ψ blows up as $x\to-\infty$) and $D=0$ (else $\psi\to\infty$ as $x\to\infty$), with B and C nonzero (so the wave function doesn't evaporate entirely). It follows [Eq. (6)] that $M_{11}=0$. Thus bound states in the locally periodic system are signaled by the vanishing of w_N,

$$we^{-iks}U_{N-1}(\xi)-U_{N-2}(\xi)=0, \tag{46}$$

at imaginary values of k.[17]

D. End conditions

1. Array in a box

Suppose, now, that the entire array is placed in an infinite square well (Fig. 5), so the wave function goes to zero at the two walls ($x=-l-s$ to the left, and $x=Ns+r$ to the right):

$$\begin{aligned} &A_0e^{-ik(l+s)}+B_0e^{ik(l+s)}=0,\\ &A_Ne^{ikr}+B_Ne^{-ikr}=0. \end{aligned} \tag{47}$$

From Eqs. (23) and (27) we have

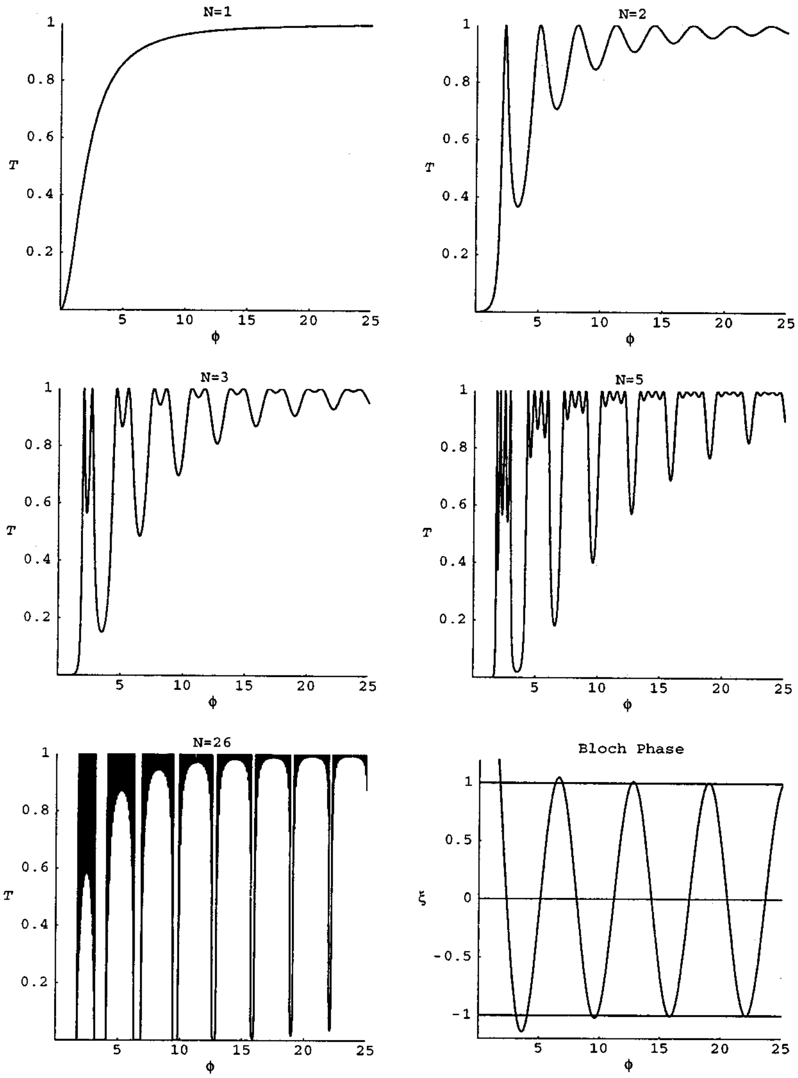

Fig. 4. Transmission coefficients for the periodic δ potential. The horizontal axis is $\phi = ks = s\sqrt{2mE}/\hbar$, and we used $mcs = 2\hbar^2$. In the last figure $\xi = \cos\gamma$, where γ is the Bloch phase.

$$\begin{aligned} A_0 &= [we^{-iks}U_{N-1} - U_{N-2}]A_N + ze^{iks}U_{N-1}B_N, \\ B_0 &= z^* e^{-iks}U_{N-1}A_N + [w^* e^{iks}U_{N-1} - U_{N-2}]B_N. \end{aligned} \tag{48}$$

Using the last three equations to eliminate A_0, B_0, and B_N, the first delivers an implicit formula for the allowed energies; after some trigonometric manipulation this reduces to

$$\begin{aligned} U_N(\xi)\sin[k(r+l+s)] = U_{N-1}(\xi)\{&|w|\sin[k(r+l)+\theta_w] \\ &-|z|\sin[k(r-l)+\theta_z]\}, \end{aligned} \tag{49}$$

where θ_w and θ_z are the phases of w and z, respectively:

$$w = |w|e^{i\theta_w}, \quad z = |z|e^{i\theta_z}. \tag{50}$$

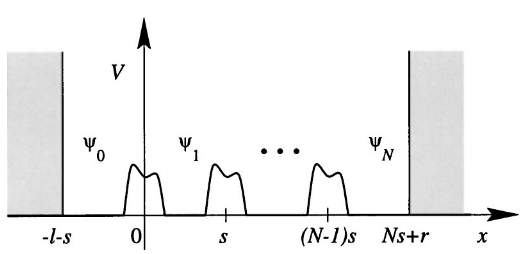

Fig. 5. Infinite square well with N arbitrary potentials.

If the array is centered in the well, so that the left and right gaps are equal ($l = r$), Eq. (49) simplifies further:

$$\begin{aligned} U_N(\xi)\sin[k(2r+s)] = U_{N-1}(\xi)\{&\cos(2kr)\mathrm{Im}(w) \\ &+\sin(2kr)\mathrm{Re}(w) - \mathrm{Im}(z)\}. \end{aligned} \tag{51}$$

For instance, if the unit cell is a delta function [Eq. (36)], and the left/right gaps are equal to the cell spacing ($l = r = 0$),[18] the condition for the allowed energies is just

$$U_N(\xi)\sin(ks) = 0. \tag{52}$$

Either $\sin(ks)=0$, in which case the wave function vanishes at each delta function, and we recover an unperturbed eigenstate of the infinite square well, or else $U_N(\xi) = 0$, which [in view of Eq. (33)] means $\sin(N+1)\gamma = 0$, or $(N+1)\gamma = n\pi$, and hence [Eqs. (34) and (38)]

$$\cos(ks) + \beta\sin(ks) = \cos\left(\frac{n\pi}{N+1}\right) \quad (n = 1,2,3,\ldots,N). \tag{53}$$

These results are illustrated in Fig. 6 for $N = 0$ (the unadorned infinite square well), $N = 1$, $N = 2$, and $N = 3$. Notice that every $(N+1)$th level is unperturbed [these are the ones that come from $\sin(ks) = 0$]. For $N = 25$ (Fig. 7) the emerging

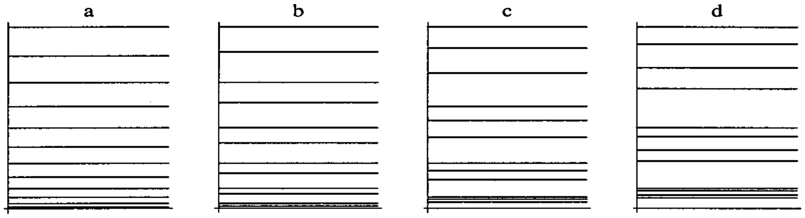

Fig. 6. Energy levels for an infinite square well with (a) no δ potentials, (b) one δ potential, (c) two δ potentials, and (d) three δ potentials. We used $mcs = 4\hbar^2$.

band structure is clearly visible, as the intermediate levels are squeezed in next to the unperturbed ones.[19]

2. Periodic boundary conditions

We recover the Kronig–Penney model itself by joining the tail of the array to its head (forming a true periodic system):

$$\psi_N(x+Ns) = \psi_0(x), \tag{54}$$

which is to say

$$A_N e^{ikx} + B_N e^{-ikx} = A_0 e^{ikx} + B_0 e^{-ikx},$$

or

$$\begin{pmatrix} A_0 \\ B_0 \end{pmatrix} = \begin{pmatrix} A_N \\ B_N \end{pmatrix},$$

and hence [Eq. (23)]

$$\mathbf{P}^N = \mathbf{I}. \tag{55}$$

From Eqs. (22) and (27) it follows that

$$we^{-iks} U_{N-1}(\xi) - U_{N-2}(\xi) = 1, \quad ze^{iks} U_{N-1}(\xi) = 0,$$

so

$$U_{N-1}(\xi) = 0, \quad U_{N-2}(\xi) = -1,$$

or [Eq. (33)]

$$\frac{\sin(N\gamma)}{\sin\gamma} = 0, \quad \frac{\sin(N-1)\gamma}{\sin\gamma} = -1,$$

where $\gamma = \cos^{-1}\xi$. The first of these yields $N\gamma = \pi l$, for some integer l; the second says

$$\frac{\sin(N\gamma)\cos\gamma - \cos(N\gamma)\sin\gamma}{\sin\gamma} = -1,$$

or $\cos(N\gamma) = 1$, so $\cos(\pi l) = 1$, and hence l must in fact be an *even* integer. It follows [Eqs. (29) and (34)] that

$$\cos\left(\frac{2\pi j}{N}\right) = \mathrm{Re}(w)\cos(ks) + \mathrm{Im}(w)\sin(ks)$$

$$\text{with } j = 1,2,3,\ldots,N. \tag{56}$$

This equation determines the allowed energies of the system and yields the familiar band structure for a periodic lattice.

The wave functions defined in Eq. (19) do not, in general, satisfy Bloch's theorem, but it is possible to choose a basis set that *does* (in the potential region $-a \leqslant x \leqslant Ns + a$). Let

$$\begin{pmatrix} A_n \\ B_n \end{pmatrix} = D \begin{pmatrix} \bar{A}_n \\ \bar{B}_n \end{pmatrix}, \tag{57}$$

where

$$\mathbf{D} = \begin{pmatrix} ze^{iks} & ze^{iks} \\ (e^{i\gamma} - we^{-iks}) & (e^{-i\gamma} - we^{-iks}) \end{pmatrix} \tag{58}$$

is the matrix that diagonalizes $\mathbf{P}$ (see Ref. 10). Then

$$\begin{pmatrix} \bar{A}_n \\ \bar{B}_n \end{pmatrix} = \begin{pmatrix} e^{i\gamma} & 0 \\ 0 & e^{-i\gamma} \end{pmatrix} \begin{pmatrix} \bar{A}_{n+1} \\ \bar{B}_{n+1} \end{pmatrix}, \tag{59}$$

and the ''upper'' state ($\bar{B}_n = 0$ for all n) satisfies

$$\psi_{n+1}(x+s) = e^{-i\gamma}\psi_n(x), \tag{60}$$

which is Bloch's theorem with phase $-\gamma$, while the ''lower'' state ($\bar{A}_n = 0$) satisfies

$$\psi_{n+1}(x+s) = e^{i\gamma}\psi_n(x), \tag{61}$$

which is Bloch's theorem with phase γ.

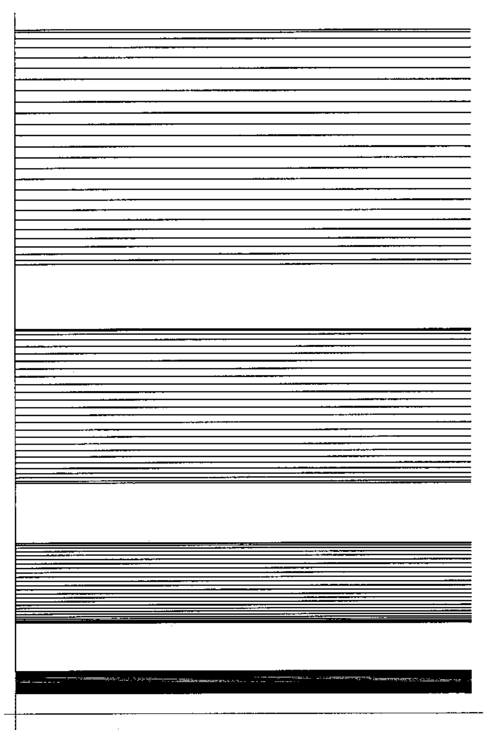

Fig. 7. Energy levels for 25 delta functions in the infinite square well.

III. MECHANICAL WAVES

In this section we explore four applications in classical mechanics:[20] transverse waves on weighted strings, longitudinal waves on loaded rods, acoustic waves in corrugated tubes, and water waves crossing a series of sandbars. Wherever possible we borrow the terminology and results from Sec. II. In particular, we continue to use the complex notation, with the understanding, always, that the *physical* wave is the real part. For example, if

$$\Psi(x,t) = Ae^{ikx}e^{-i\omega t}, \tag{62}$$

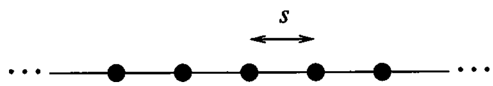

Fig. 8. A periodically weighted string.

with the complex amplitude

$$A=|A|e^{i\theta}, \tag{63}$$

the actual wave is

$$\mathrm{Re}(\Psi)=|A|\cos(kx-\omega t+\theta), \tag{64}$$

with (real) amplitude $|A|$ and phase constant θ.

A. Transverse waves on a weighted string

Imagine a uniform taut string, infinitely long (or long enough, at any rate, so that we don't have to worry about waves reflected from the ends). In the central portion we attach N equal weights, at regular intervals s (Fig. 8)—or, if you prefer, we splice in N identical segments of greater (or lesser, or varying) density.[21] If we now shake the (distant) left end sinusoidally, at angular frequency ω, a wave propagates down the line, and we would like to know how much is transmitted (and how much reflected) when it encounters the weights.

For small oscillations, transverse displacements $\Psi(x,t)$ of the string are governed by the classical wave equation,

$$\frac{\partial^2\Psi}{\partial t^2}=v^2\frac{\partial^2\Psi}{\partial x^2}, \tag{65}$$

with propagation speed

$$v(x)=\sqrt{T/\mu(x)}, \tag{66}$$

where T is the tension and $\mu(x)$ is the linear mass density. Separable solutions take the form

$$\Psi(x,t)=\psi(x)e^{-i\omega t}, \tag{67}$$

with

$$\frac{d^2\psi}{dx^2}=-k^2\psi,$$

where

$$k(x)\equiv\frac{\omega}{v(x)}. \tag{68}$$

Within each cell the mass density $\mu(x)$ and hence also $v(x)$ and $k(x)$ are functions of x, but we shall reserve the letter k (with no argument) for the constant ambient value on the open portions of the string.

For a single cell the solution is exactly the same as before [Eq. (4)]; the only difference is that the function ψ_{ab} now satisfies Eq. (68) (and the attendant boundary conditions) instead of the Schrödinger equation. The time reversal argument runs the same as before [complex conjugation is necessary to restore the canonical form of the time dependence, Eq. (67)]. Conservation of probability becomes conservation of *energy*, but the algebraic consequence is the same [Eq. (13)]. Thus the transfer matrix retains the generic structure of Eq. (17), and all the machinery of Sec. II carries over.

For example, suppose the unit cell consists of a single point mass m at $x=0$. Then

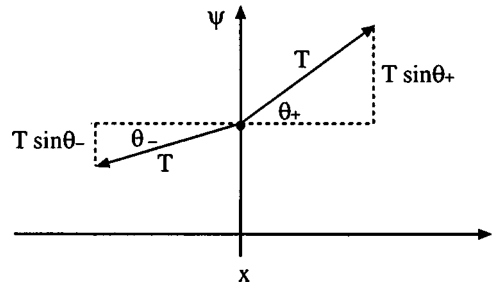

Fig. 9. The transverse force on the weight is $T(\sin\theta_+-\sin\theta_-)$.

$$\Psi(x,t)=\begin{cases}Ae^{ikx}e^{-i\omega t}+Be^{-ikx}e^{-i\omega t} & \text{if } x<0,\\ Ce^{ikx}e^{-i\omega t}+De^{-ikx}e^{-i\omega t} & \text{if } x>0.\end{cases} \tag{69}$$

Continuity at the join implies

$$A+B=C+D. \tag{70}$$

Meanwhile, the net (transverse) force on m is (Fig. 9)

$$T\sin\theta_+-T\sin\theta_-\cong T\Delta\left(\frac{\partial\Psi}{\partial x}\right) \tag{71}$$

(where Δ denotes the change in the quantity that follows), and Newton's second law gives

$$T\Delta\left(\frac{\partial\Psi}{\partial x}\right)=m\frac{\partial^2\Psi}{\partial t^2}. \tag{72}$$

According to Eq. (69), then,

$$Tik(C-D-A+B)=-m\omega^2(C+D). \tag{73}$$

Solving Eqs. (70) and (73) for A and B in terms of C and D, we find

$$\begin{pmatrix}A\\B\end{pmatrix}=\begin{pmatrix}1-i\chi & -i\chi\\ i\chi & 1+i\chi\end{pmatrix}\begin{pmatrix}C\\D\end{pmatrix}, \tag{74}$$

where $\chi\equiv m\omega^2/2kT$. This is identical to the transfer matrix for quantum scattering from a delta function function [Eq. (37)], with $\beta\to-\chi$,[22] and we can immediately read off the transmission coefficient for an array of N such masses [Eq. (39)]:

$$T_N=\frac{1}{1+[\chi U_{N-1}(\xi)]^2}, \tag{75}$$

where [Eq. (38)]

$$\xi=\cos(ks)-\chi\sin(ks). \tag{76}$$

In this context, of course, T_N does not refer to the probability of transmission, but rather to the fraction of the energy transmitted. If you wanted, for some reason, to design a string that would not transmit waves in a certain frequency range, you could attach weights in such a way that the excluded range falls into one of the ''gaps'' where $T\to 0$ (Fig. 4).

Of course, infinite strings are expensive, and awkward to work with in the laboratory, but it is easy to test these results using an ordinary standing-wave apparatus with a *finite* string nailed down at the two ends. This is analogous to putting the quantum system in an infinite square well, and we can simply quote our previous results (Sec. II D 1).[23]

The analog of the rectangular barrier (Sec. II C 2) is a segment of string with a different (constant) mass density, μ'.

Since the boundary conditions are the same as before (continuity of ψ and $d\psi/dx$), the solution is unchanged [Eqs. (40)–(43)], with

$$k=\omega\sqrt{\frac{\mu}{T}},\quad k'=\omega\sqrt{\frac{\mu'}{T}}. \tag{77}$$

B. Longitudinal waves on a loaded rod

Next consider compressional waves on a long uniform rod. Let $\Psi(x,t)$ be the displacement of a point whose equilibrium position is x. Newton's second law, applied to a segment of length Δx, says[24]

$$\frac{\partial F}{\partial x}\Delta x=S\rho\Delta x\frac{\partial^2\Psi}{\partial t^2}, \tag{78}$$

where F is the tensile force, S the cross-sectional area, and ρ the mass density. For small disturbances the stress (F/S) is proportional to the strain ($\partial\Psi/\partial x$):

$$\frac{F}{S}=Y\frac{\partial\Psi}{\partial x}, \tag{79}$$

where Y is Young's modulus for the material. Combining Eqs. (78) and (79) we find that (in regions where S and Y are constant) Ψ satisfies the classical wave equation (65), with propagation speed

$$v=\sqrt{Y/\rho}. \tag{80}$$

(To reduce the speed one can replace the rod with a spring; if the spring constant for a segment of length L is K, then $Y\to KL/S$, and $v=\sqrt{KL/\mu}$, where μ is the mass per unit length.)

Now imagine loading the rod by splicing in a sequence of N identical cells of varying area and composition, so that Y, ρ, and/or S are locally periodic functions of x (in the spring case we would vary the stiffness and/or the mass per unit length).[25] If the unit cell consists of a single point mass m embedded in the rod at point $x=0$, then [using Newton's second law and Eq. (79)]:

$$m\frac{\partial^2\Psi}{\partial x^2}=\Delta F=SY\Delta\left(\frac{\partial\Psi}{\partial x}\right), \tag{81}$$

which is the same (apart from the constants) as Eq. (72) for the transverse case, and we recover Eq. (74), with $\chi=m\omega^2/2kSY$ (or, for a spring, $\chi=m\omega^2/2kKL$). Again, we have the analog to quantum scattering from a delta function well.

If the unit cell consists of a uniform segment (of length $2a$) with parameters ρ', Y', and S', the boundary conditions are (1) $\Psi(x,t)$ continuous, and (2) $F(x,t)$ continuous (otherwise there would be a net force on a point of zero mass). Therefore [Eq. (79)]

$$\Delta\left(SY\frac{\partial\Psi}{\partial x}\right)=0. \tag{82}$$

This time it is not simply the derivative of Ψ that is continuous; the transfer matrix (41) is unaffected, but in the definition of η [Eq. (42)], k is multiplied by SY and k' by $S'Y'$.

If S and Y vary continuously, the situation is more complicated. Equations (78) and (79) yield a modified wave equation

$$\frac{\partial^2\Psi}{\partial t^2}=\frac{Y}{\rho}\frac{\partial^2\Psi}{\partial x^2}+\frac{1}{S\rho}\frac{d(SY)}{dx}\frac{\partial\Psi}{\partial x}. \tag{83}$$

If substantial variations occur only over a scale large compared to the wavelength, then the second term on the right is negligible, and the waves simply propagate with a speed that depends on x [Eq. (80)]. But if the variations are significant on a scale comparable to (or less than) the wavelength, then the results depend on the functional forms of $S(x)$ and $Y(x)$. We'll see some examples in Sec. III C 2.

C. Acoustics

1. Plane waves

Sound propagation in a perfect fluid satisfies the classical wave equation; in one dimension,

$$\frac{\partial^2\Psi}{\partial t^2}=v^2\frac{\partial^2\Psi}{\partial x^2}, \tag{84}$$

where $\Psi(x,t)$ now represents the pressure above ambient. For an ideal gas the wave speed is

$$v=\sqrt{\gamma RT_0}, \tag{85}$$

where T_0 is the (ambient) temperature, R is the gas constant, and γ is the ratio of the specific heats.[26] It is not easy to conceive of a realistic system in which the relevant parameters (γ and/or T_0) vary in a locally periodic manner,[27] but we can set up baffles to force locally periodic boundary conditions.

Suppose a monochromatic sound wave encounters a pane of glass, at normal incidence; as always,

$$\Psi(x,t)=\begin{cases}Ae^{ikx}e^{-i\omega t}+Be^{-ikx}e^{-i\omega t} & \text{if } x<0\\ Ce^{ikx}e^{-i\omega t}+De^{-ikx}e^{-i\omega t} & \text{if } x>0,\end{cases} \tag{86}$$

with $k=\omega/v$. (We assume the window is thin, compared to the wavelength, and can simply be treated as a heavy layer at $x=0$, with mass-per-unit-area σ.) Newton's second law says

$$\Delta\Psi=-\sigma\frac{\partial u}{\partial t}, \tag{87}$$

where u is the velocity of the glass (which is also the velocity of the gas on either side); it is related to the pressure:[28]

$$\frac{\partial\Psi}{\partial x}=-\rho_0\frac{\partial u}{\partial t}, \tag{88}$$

where ρ_0 is the ambient gas density (which we shall assume is the same on both sides). Thus the boundary conditions at the window are

$$\Delta\left(\frac{\partial\Psi}{\partial x}\right)=0,\quad \Delta\Psi=\frac{\sigma}{\rho_0}\left(\frac{\partial\Psi}{\partial x}\right). \tag{89}$$

This yields a transfer matrix reminiscent (except for the signs) of the delta-function barrier (Sec. II C 1) and the string loaded with a point mass (Sec. III A):

$$\begin{pmatrix}A\\B\end{pmatrix}=\begin{pmatrix}1-i\chi & i\chi\\ -i\chi & 1+i\chi\end{pmatrix}\begin{pmatrix}C\\D\end{pmatrix}, \tag{90}$$

where $\chi\equiv\sigma k/2\rho_0$. For a succession of N such windows, a distance s apart, the transmission coefficient would be

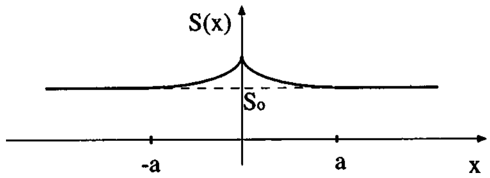

Fig. 10. Double exponential horn [Eq. (103)]; $S(x)$ is the cross-sectional area.

$$T_N=\frac{1}{1+[\chi U_{N-1}(\xi)]^2}, \tag{91}$$

with

$$\xi=\cos(ks)-\chi\sin(ks). \tag{92}$$

Such an array can be used as an acoustic filter, transmitting sound in the "allowed" frequency ranges and rejecting it in the "forbidden" gaps.[29]

2. Sound waves in pipes

Sound waves confined to a tube satisfy an equation identical in structure to the one describing longitudinal waves in an elastic rod [Eq. (83)].[30] In this context it is known as the "horn equation," in recognition of its most familiar application:

$$\frac{\partial^2\Psi}{\partial t^2}=v^2\left(\frac{\partial^2\Psi}{\partial x^2}+\frac{1}{S}\frac{dS}{dx}\frac{\partial\Psi}{\partial x}\right). \tag{93}$$

Here v is again the speed of sound [Eq. (85)] and S is the cross-sectional area of the tube.[31] If S is constant, we recover the classical wave equation, and the conditions of Sec. III C 1. For variable cross section there are precious few cases that can be solved analytically.[32] The simplest of these is the "exponential horn,"

$$S(x)=S(0)e^{2\Gamma x}, \tag{94}$$

for which Eq. (93) reduces to

$$\frac{\partial^2\Psi}{\partial t^2}=v^2\left(\frac{\partial^2\Psi}{\partial x^2}+2\Gamma\frac{\partial\Psi}{\partial x}\right). \tag{95}$$

For monochromatic waves, $\Psi(x,t)=\psi(x)\exp(-i\omega t)$, and the spatial wave function satisfies the damped harmonic oscillator equation:

$$\frac{d^2\psi}{dx^2}+2\Gamma\frac{d\psi}{dx}+k^2\psi=0 \tag{96}$$

(with $k=\omega/v$, as always). The general solution is

$$\psi(x)=e^{-\Gamma x}(A'e^{ik'x}+B'e^{-ik'x}), \tag{97}$$

where

$$k'\equiv\sqrt{k^2-\Gamma^2}. \tag{98}$$

Now imagine a long tube, of constant cross section S_0 joined to an expanding exponential horn at $x=-a$, followed symmetrically by a contracting horn from $x=0$ out to $x=a$ (Fig. 10):

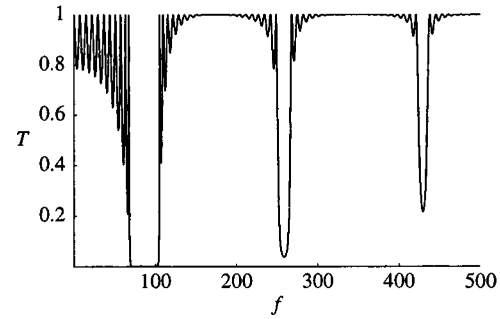

Fig. 11. Transmission coefficient for ten exponential corrugations ($\gamma=0.5$, $a=1$, $s=2$, frequency f in hertz).

$$S(x)=\begin{cases} S_0 & \text{if } x<-a \\ S_0e^{2\Gamma(a+x)} & \text{if } -a<x<0 \\ S_0e^{2\Gamma(a-x)} & \text{if } 0<x<a \\ S_0 & \text{if } x>a. \end{cases} \tag{99}$$

Then

$$\psi(x)=\begin{cases} Ae^{ikx}+Be^{-ikx} & \text{if } x<-a \\ e^{-\Gamma x}(A'e^{ik'x}+B'e^{-ik'x}) & \text{if } -a<x<0 \\ e^{\Gamma x}(A''e^{ik'x}+B''e^{-ik'x}) & \text{if } 0<x<a \\ Ce^{ikx}+De^{-ikx} & \text{if } x>a. \end{cases} \tag{100}$$

This time Ψ and u are continuous, so [Eq. (88)]

$$\Delta\psi=0, \quad \Delta\left(\frac{d\psi}{dx}\right)=0 \tag{101}$$

at each boundary, and (after some algebra) we obtain the elements of the transfer matrix:

$$\begin{aligned} w&=\frac{e^{2ika}}{(k')^2}[k^2\cos(2k'a)-\Gamma^2-ikk'\sin(2k'a)], \\ z&=-i\frac{2\Gamma k}{(k')^2}\sin^2(k'a). \end{aligned} \tag{102}$$

We can now immediately construct the transmission coefficient for a tube with N such exponential corrugations, using Eqs. (29) and (30). Figure 11 shows a typical graph, for $N=10$.

Of course, exponential corrugation is rather artificial; we used it only as an illustration. It is possible to treat tubes with rectangular corrugations, though the method is necessarily less rigorous. The horn equation (93) presupposes that any changes in cross section are gradual. In the vicinity of an *abrupt* change the waves are no longer even one dimensional, as they "diffract" around the edge. Still, at frequencies well below cutoff the transverse modes are rapidly attenuated, and the non-plane-wave zone is short enough that it can be ignored.[33] In this low-frequency regime the pressure is a continuous function of x, while the continuity equation (expressing conservation of mass) requires that the product Su be continuous; the boundary conditions are therefore

$$\Delta\psi=0, \quad \Delta\left(S\frac{d\psi}{dx}\right)=0. \tag{103}$$

For a rectangular bulge (Fig. 12)

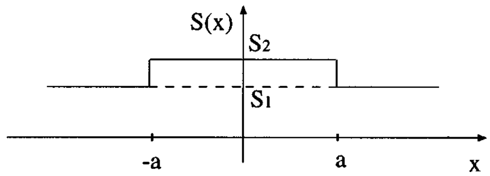

Fig. 12. Rectangular corrugation; $S(x)$ is the cross-sectional area.

$$\psi(x)=\begin{cases} Ae^{ikx}+Be^{-ikx} & \text{if } x<-a \\ Fe^{ikx}+Ge^{-ikx} & \text{if } -a<x<a \\ Ce^{ikx}+De^{-ikx} & \text{if } x>a. \end{cases} \tag{104}$$

(In this case $k=\omega/v$ is the same in all three regions.) Invoking the boundary conditions [Eq. (103)], we obtain the transfer matrix elements

$$\begin{aligned} w&=[\cos(2ka)-i\varepsilon_+\sin(2ka)]e^{2ika}, \\ z&=i\varepsilon_-\sin(2ka), \end{aligned} \tag{105}$$

the same as Eq. (41) (with $\varepsilon_\pm\equiv(1/2)[\eta\pm(1/\eta)]$), except that this time

$$\eta\equiv\frac{S_1}{S_2}. \tag{106}$$

For N corrugations (separated by a distance s), the transmission coefficient is [Eq. (32)]

$$T_N=\frac{1}{1+[\varepsilon_-\sin(2ka)U_{N-1}(\xi)]^2}, \tag{107}$$

where [Eq. (29)]

$$\xi=\cos(2ka)\cos(k\ell)-\varepsilon_+\sin(2ka)\sin(k\ell), \tag{108}$$

with $\ell\equiv s-2a$, as before.

These results are similar in form to quantum scattering from a rectangular barrier (Sec. II C 2), but the band structure is quite different, because η (and hence $\varepsilon_\pm$) are now *constants*, independent of the frequency. For example, if the spacing between corrugations is the same as their width ($s=4a$), then

$$\xi=1-(1+\varepsilon_+)\sin^2(2ka). \tag{109}$$

Suppose the radii are 4 and 5 cm, and $a=2.5$ cm. In that case the cutoff frequency (above which our analysis fails) is around 4000 Hz. In Fig. 13 we plot the transmission coeffi-

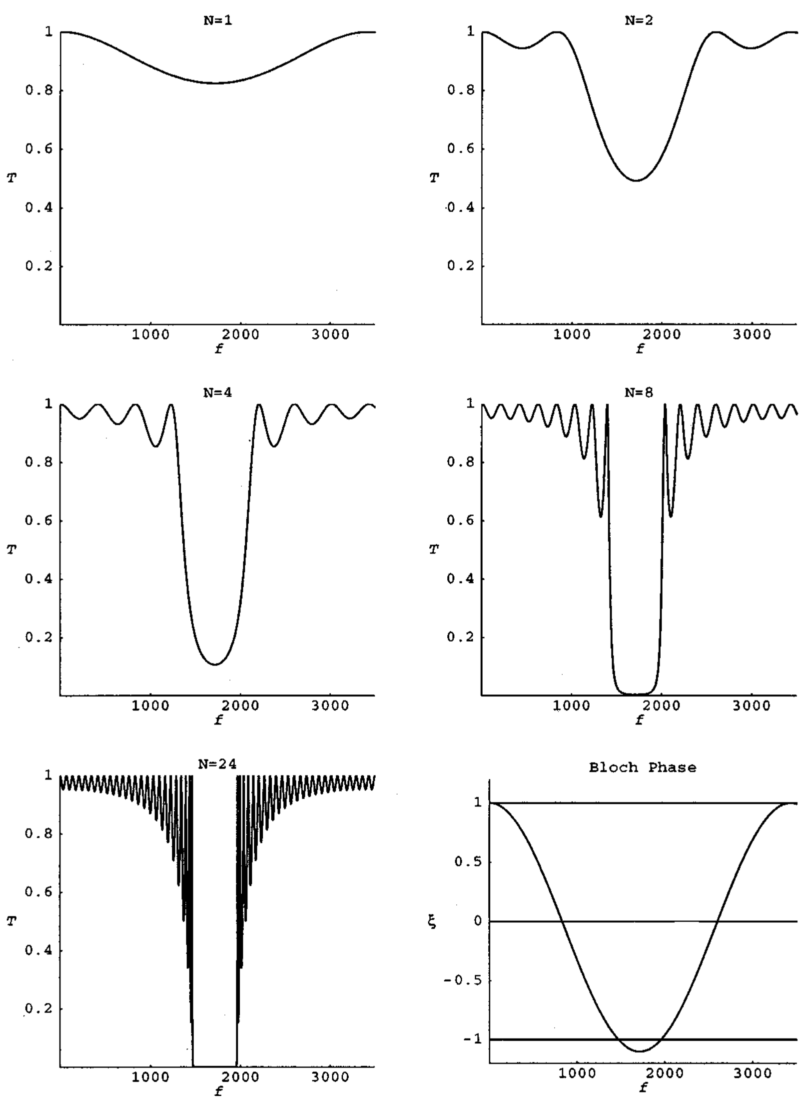

Fig. 13. Corrugated tube acoustic filter. In the last figure $\xi=\cos\gamma$, where γ is the Bloch phase.

n	k_n	f_n (Hz)	Smooth
1	3.3683	182	187
2	6.7367	364	374
3	10.1049	547	560
4	13.4732	729	747
5	16.8413	911	934
6	20.2094	1093	1121
7	23.5774	1276	1308
8	26.9452	1458	1495

Fig. 14. Calculated resonant frequencies for Crawford's Hummer.

cients for $N=1$, 2, 4, 8, and 24, as well as the Bloch phase. The striking feature is of course the central gap (from around 1500 to 2000 Hz), in which no sound is transmitted. Such acoustic filters are used in mufflers, gun silencers, and ventilation systems.[34]

This system is reminiscent of Crawford's "corrugahorn."[35] Crawford was concerned with an entirely different phenomenon (stimulation of resonances by blowing air through the tube), but he noted in passing that the fundamental was "about 4%" lower in pitch than it would be for a smooth pipe of the same overall length and diameter—an effect he attributed to the "extra" length resulting from the corrugations. Using the method of Sec. II D 1 we are now in a position to *calculate* the resonant frequencies of a corrugahorn. For a completely corrugated tube ($l=r=0$), with spacing equal to width ($s=4a$), the resonance condition [Eq. (51)] becomes

$$2\sin(2ka)\cos(2ka)U_N(\xi) = \sin(2ka)[(1-\varepsilon_+)\cos(2ka)-\varepsilon_-]U_{N-1}(\xi), \tag{110}$$

so either

$$\sin(2ka)=0, \tag{111}$$

or else

$$2\cos(2ka)U_N(\xi)=[(1-\varepsilon_+)\cos(2ka)-\varepsilon_-]U_{N-1}(\xi) \tag{112}$$

[with ξ given by Eq. (109)]. The former yields wavelengths $\lambda=4a/n$ ($n=1,2,3,...$); there is a node at each edge, so these modes are unperturbed by the corrugations. The fundamental, however, corresponds to the smallest k that satisfies Eq. (112).[36]

Crawford says his "Hummer" is $L=3$ ft$=91$ cm long, the corrugations are $s=0.64$ cm apart, and the radius ranges from $r_1=1.2$ cm to $r_2=1.5$ cm.[37] Evidently $N=L/s-1=141$, $a=s/4=0.16$ cm, $\varepsilon_+=(1/2)[(r_1/r_2)^2+(r_2/r_1)^2]=1.10$, and $\varepsilon_-=(1/2)[(r_1/r_2)^2-(r_2/r_1)^2]=-0.46$. So

$$\xi=1-(2.10)\sin^2(0.32k) \tag{113}$$

[Eq. (109)], and the resonance condition (112) is

$$2\cos(0.32k)U_{141}(\xi)=[-0.10\cos(0.32k)+0.46]U_{140}(\xi). \tag{114}$$

Roots to this equation determine k_n (whence $f_n=vk_n/2\pi$). Results are shown in Fig. 14 (we used 340 m/s for the speed of sound). In particular, $f_1=182$ Hz, as compared with a "smooth pipe" value of $v/2L=187$ Hz.[38] This confirms Crawford's observation that corrugation suppresses the fundamental, and the theoretical factor (3%) is reasonably close to his empirical estimate (4%).

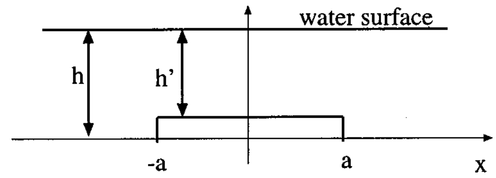

Fig. 15. A shoal or sandbar.

D. Water waves

1. Shallow waves and sandbars

In the case of water waves, we let (the real part of) $\Psi(x,t)$ represent the height of the surface above its equilibrium level. For small displacements, sinusoidal waves,

$$\Psi(x,t)=\psi(x)e^{-i\omega t}$$

with

$$\psi(x)=Ae^{ikx}+Be^{-ikx}, \tag{115}$$

propagate in regions of constant depth h at a speed[39]

$$v=\frac{\omega}{k}=\sqrt{\frac{g}{k}\tanh(kh)}, \tag{116}$$

where g is the acceleration due to gravity. In "shallow" water, where $h/\lambda=hk/2\pi<0.05$, Eq. (116) reduces to

$$v=\sqrt{gh}, \tag{117}$$

and the waves are nondispersive (the speed is independent of frequency); in "deep" water, where $h/\lambda>0.5$, Eq. (116) becomes

$$v=\sqrt{\frac{g}{k}}, \tag{118}$$

and the waves are dispersive, but insensitive to depth. The intermediate regime, $0.05<h/\lambda<0.5$, is more complicated. We shall restrict our attention to shallow water waves.[40]

Suppose such a wave [Eq. (115)] encounters a shoal or sandbar, where the depth changes abruptly from h to h' (Fig. 15). This is analogous to the rectangular barrier in Sec. II C 2; $\psi(x)$ is given by Eq. (40), with $k=\omega/\sqrt{gh}$ and $k'=\omega/\sqrt{gh'}$. The boundary conditions are[41]

$$\Delta\psi=0, \quad \Delta\left(h\frac{d\psi}{dx}\right)=0, \tag{119}$$

and the transfer matrix is again given by Eq. (41), except that in the definition of η [Eq. (42)], k must be multiplied by h, and k' by h'. The transmission coefficient for a series of sandbars is given, as always, by Eq. (32), with the variables defined in Eq. (43). Such an array would prevent the passage of waves in the forbidden frequency bands.[42]

For slowly varying depths, surface waves satisfy[43]

$$\frac{d^2\psi}{dx^2}+\frac{1}{h}\frac{dh}{dx}\frac{d\psi}{dx}+k^2\psi=0. \tag{120}$$

This is the time-independent form of Webster's horn equation [Eq. (93)], and the results of Sec. III C 2 carry over directly.

2. Water waves in canals

Water waves in a narrow channel are rather like sound waves in a tube. At low frequencies they are essentially one dimensional. We can provide for local periodicity either by modulating the depth of the channel $h(x)$ or by varying its width $w(x)$. The continuity equation says

$$w\frac{\partial\Psi}{\partial t}=-\frac{\partial}{\partial x}(whu), \tag{121}$$

where $u(x,t)$ is the horizontal component of the velocity,[44] given by

$$\frac{\partial u}{\partial t}=-g\frac{\partial\Psi}{\partial x}. \tag{122}$$

If the width is constant, the theory is the same as for shallow ocean waves (Sec. III D 1); if the depth is constant, we recover Webster's horn equation [Eq. (93), with $w(x)$ in place of $S(x)$], for continuous variation, and boundary conditions (103) (again, with w in place of S) for abrupt changes.[45] In either case the theory proceeds exactly as before.

The water–wave analog to the corrugahorn would be a long narrow tank with varying width and/or depth. The boundary condition at the ends is $\partial\Psi/\partial x=0$, instead of $\Psi=0$, but that merely shifts the phase. The spectrum of normal modes should exhibit the familiar quasiband structure, but as far as we know this has not been tested in the laboratory.[46] A related geophysical phenomenon is ''sloshing,'' or ''seiches,'' in long narrow bays and lakes.[47]

IV. ELECTROMAGNETIC WAVES

A. Transmission lines

A transmission line consists of two very long parallel conductors. Various geometries are commonly used—coaxial cables, paired wires, separated ribbons—and the space between the conductors is typically filled with insulating material. Transmission lines are conveniently analyzed in terms of ''distributed'' circuit elements: $\mathcal{C}$, the capacitance per unit length, and $\mathcal{L}$, the inductance per unit length (we shall consider resistanceless lines only). The voltage difference between the conductors, $V(x,t)$, and the current in each, $I(x,t)$ for one and $-I(x,t)$ for the other, satisfy[48]

$$\frac{\partial V}{\partial x}=-\mathcal{L}\frac{\partial I}{\partial t},\quad \frac{\partial I}{\partial x}=-\mathcal{C}\frac{\partial V}{\partial t}. \tag{123}$$

Differentiating, to separate the variables, we obtain the classical wave equation,

$$\frac{\partial^2\Psi}{\partial t^2}=v^2\frac{\partial^2\Psi}{\partial x^2}, \tag{124}$$

where $\Psi(x,t)$ represents either V or I, and the speed of propagation is[49]

$$v=\frac{1}{\sqrt{\mathcal{LC}}}. \tag{125}$$

The current and the voltage are both continuous at a junction between one transmission line and another, so the boundary conditions on Ψ are

$$\Delta\Psi=0,\quad \Delta\left(\frac{\partial\Psi}{\partial x}\right)=0. \tag{126}$$

This system is mathematically identical to the rectangular barrier (Sec. II C 2); the transfer matrix (if we insert a segment with different capacitance and inductance) is given by Eq. (41), with

$$k=\frac{\omega}{v}=\omega\sqrt{\mathcal{LC}},\quad k'=\frac{\omega}{v'}=\omega\sqrt{\mathcal{L}'\mathcal{C}'}. \tag{127}$$

The transmission coefficient for N identical segments follows immediately from Eq. (32). Such a transmission line will freely pass signals in the allowed frequency bands, but is essentially nonconducting in the forbidden gaps.

B. Layered optical media

Consider now a plane monochromatic electromagnetic wave, propagating through a homogeneous linear material of permittivity ϵ and permeability μ:[50]

$$\begin{aligned}\mathbf{E}(x,t)&=(Ae^{ikx}+Be^{-ikx})e^{-i\omega t}\hat{\mathbf{y}},\\ \mathbf{B}(x,t)&=\frac{1}{v}(Ae^{ikx}-Be^{-ikx})e^{-i\omega t}\hat{\mathbf{z}},\end{aligned} \tag{128}$$

where (again) $k=\omega/v$ and $v=1/\sqrt{\epsilon\mu}$. Here (the real part of) $\mathbf{E}$ is the electric field, and (the real part of) $\mathbf{B}$ is the magnetic field; the wave is polarized in the y direction, and travels in the $\pm x$ direction. To conform with our previous notation, let $\Psi(x,t)=\psi(x)e^{-i\omega t}$ be the y component of $E(x,t)$:

$$\mathbf{E}(x,t)=\psi(x)e^{-i\omega t}\hat{\mathbf{y}},\quad \mathbf{B}(x,t)=-\frac{i}{\omega}\frac{d\psi}{dx}e^{-i\omega t}\hat{\mathbf{z}}. \tag{129}$$

Now suppose this wave encounters a region (say, a pane of glass) in which ϵ and μ are different (call them ϵ' and μ'). This is the optical analog to the rectangular barrier (Sec. II C 2). At the boundaries $(\pm a)$, $\mathbf{E}^{\parallel}$ and $(1/\mu)\mathbf{B}^{\parallel}$ are continuous, so

$$\Delta\psi=0,\quad \Delta\left(\frac{1}{\mu}\frac{d\psi}{dx}\right)=0. \tag{130}$$

The transfer matrix is the same as before [Eq. (41)], but this time

$$\eta\equiv\sqrt{\frac{\epsilon\mu'}{\epsilon'\mu}}. \tag{131}$$

For N layers, the transmission coefficient (representing, in this context, the fraction of the incident intensity that makes it through) is[51]

$$T_N=\frac{1}{1+[\varepsilon_-\sin(2k'z)U_{N-1}(\xi)]^2}, \tag{132}$$

with

$$\xi=\cos(2k'a)\cos(k\ell)-\varepsilon_+\sin(2k'a)\sin(k\ell), \tag{133}$$

where (as always) $\ell\equiv s-2a$. As in the quantum case (Fig. 4), the emerging band structure is apparent with surprisingly few layers. But the details are different, because (as in the acoustic analog, Fig. 13) $\varepsilon_\pm$ are now independent of ω.[52]

V. THE DIRAC EQUATION

The Dirac equation describes relativistic particles of spin 1/2 (such as the electron). In the absence of interactions, it reads[53]

$$i\hbar\gamma^\mu\partial_\mu\Psi - mc\Psi = 0, \tag{134}$$

where m is the particle's mass,

$$\partial_\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

is the four-dimensional gradient operator, and summation over the index $\mu(0\to 3)$ is implied. In block notation the (4×4) Dirac matrices γ^μ are

$$\gamma^0\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \quad \gamma^i = \begin{pmatrix}0 & \sigma_i\\ -\sigma_i & 0\end{pmatrix} \quad (i{:}1=x,2=y,3=z), \tag{135}$$

where σ_i are the (2×2) Pauli matrices,

$$\sigma_x = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \quad \sigma_y = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, \quad \sigma_z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}; \tag{136}$$

Ψ itself is a four-element spinor:

$$\Psi = \begin{pmatrix}\Psi_1\\ \Psi_2\\ \Psi_3\\ \Psi_4\end{pmatrix}. \tag{137}$$

In the presence of interactions, Eq. (134) becomes

$$i\hbar c\gamma^\mu\partial_\mu\Psi - (mc^2 + V + \gamma^\mu A_\mu)\Psi = 0, \tag{138}$$

where V is an external scalar potential and A^μ a vector potential (it is possible to introduce other kinds of interactions, of course, but we shall restrict our attention to these two).

A. Reduction to one dimension

In the one-dimensional case of interest here, $\Psi(x,y,z,t)\to\Psi(x,t)$, the y and z derivatives are zero, and the three-vector potential drops out, $A^\mu = (A^0,\mathbf{A})\to(A,\mathbf{0})$; the remaining potentials, V and A, depend only on x, and Eq. (138) reduces to

$$i\hbar\gamma^0\frac{\partial\Psi}{\partial t} + i\hbar c\gamma^1\frac{\partial\Psi}{\partial x} - (mc^2 + V + A\gamma^0)\Psi = 0. \tag{139}$$

In component form,

$$\begin{aligned}
&i\hbar\frac{\partial\Psi_1}{\partial t} + i\hbar c\frac{\partial\Psi_4}{\partial x} - (mc^2+V+A)\Psi_1 = 0,\\
&i\hbar\frac{\partial\Psi_2}{\partial t} + i\hbar c\frac{\partial\Psi_3}{\partial x} - (mc^2+V+A)\Psi_2 = 0,\\
&i\hbar\frac{\partial\Psi_3}{\partial t} + i\hbar c\frac{\partial\Psi_2}{\partial x} + (mc^2+V-A)\Psi_3 = 0,\\
&i\hbar\frac{\partial\Psi_4}{\partial t} + i\hbar c\frac{\partial\Psi_1}{\partial x} + (mc^2+V-A)\Psi_4 = 0.
\end{aligned} \tag{140}$$

A simplifying feature of the one-dimensional regime is that the four components of Ψ mix only in pairs: Ψ_1 with Ψ_4, and Ψ_2 with Ψ_3. This invites us to introduce the *two*-component spinor

$$\Psi = \begin{pmatrix}\Psi_1\\ \Psi_4\end{pmatrix}$$

or

$$\begin{pmatrix}\Psi_2\\ \Psi_3\end{pmatrix}, \tag{141}$$

in terms of which the one-dimensional Dirac equation assumes the standard form

$$i\hbar\frac{\partial\Psi}{\partial t} + i\hbar c\alpha\frac{\partial\Psi}{\partial x} - (mc^2+V)\beta\Psi - A\Psi = 0, \tag{142}$$

where $\alpha = \sigma_x$ and $\beta = \sigma_z$.[54]

As always, we are interested here in "monochromatic" waves:

$$\Psi(x,t) = \psi(x)e^{-iEt/\hbar}; \tag{143}$$

this reduces the Dirac equation to time-independent form:

$$i\hbar c\alpha\frac{d\psi}{dx} = (mc^2+V)\beta\psi = (A-E)\psi. \tag{144}$$

Calling the upper component ψ_u and the lower component ψ_l:

$$\begin{aligned}
&i\hbar c\frac{d\psi_l}{dx} = (mc^2+V+A-E)\psi_u,\\
&i\hbar c\frac{d\psi_u}{dx} = (-mc^2-V+A-E)\psi_l.
\end{aligned} \tag{145}$$

Evidently

$$\psi_l = \frac{-i\hbar c}{(E+mc^2+V-A)}\frac{d\psi_u}{dx}, \tag{146}$$

so we need only determine ψ_u. Differentiating the second part of Eq. (145) and eliminating ψ_l yield

$$\frac{d^2\psi_u}{dx^2} - \frac{1}{(E+mc^2+V-A)}\left(\frac{dV}{dx} - \frac{dA}{dx}\right)\frac{d\psi_u}{dx} + \frac{[(E-A)^2-(mc^2+V)^2]}{(\hbar c)^2}\psi_u = 0. \tag{147}$$

In particular, in regions where the potentials are zero,

$$\frac{d^2\psi_u}{dx^2} = -\frac{[(E)^2-(mc^2)^2]}{(\hbar c)^2}\psi_u, \tag{148}$$

and the general solution is

$$\psi_u(x) = Ae^{ikx} + Be^{-ikx}, \tag{149}$$

with[55]

$$k \equiv \frac{1}{\hbar c}\sqrt{E^2-(mc^2)^2}. \tag{150}$$

B. The transfer matrix

The theory of scattering proceeds much as before (Sec. II A), except that ψ_u now satisfies Eq. (147), instead of Schrödinger's equation, in the potential region $a<x<b$. It is easy to check that if $\Psi(x,t)$ is a solution to the one-dimensional Dirac equation (142), so too is the "time-reversed" spinor

$$\beta\Psi^*(x,-t). \tag{151}$$

In particular, for the free-particle solution [Eqs. (149) and (146)]

$$\psi=\begin{pmatrix} Ae^{ikx}+Be^{-ikx} \\ \frac{\hbar ck}{(mc^2+E)}(Ae^{ikx}-Be^{-ikx}) \end{pmatrix}, \tag{152}$$

we obtain

$$\beta\psi^*=\begin{pmatrix} A^*e^{-ikx}+B^*e^{ikx} \\ \frac{-\hbar ck}{(mc^2+E)}(A^*e^{-ikx}-B^*e^{ikx}) \end{pmatrix}. \tag{153}$$

Thus the transcription $(A,B)\to(B^*,A^*)$ takes solutions into solutions, as in the nonrelativistic case (Sec. II A), and the transfer matrix again satisfies Eq. (11). Meanwhile, the current

$$j(x)=c\psi^\dagger\alpha\psi \tag{154}$$

is conserved $(dj/dx=0)$, as one can quickly show using Eq. (144). For the free particle solution [Eq. (152)],

$$j=\frac{2\hbar c^2k}{mc^2+E}(|A|^2-|B|^2), \tag{155}$$

so Eq. (13) holds as well, and the transfer matrix takes the same generic form [Eqs. (17) and (18)] as before. That's all we need to recover the *N*-cell solution [Eq. (30)]. Moreover, Eq. (155) indicates that (even though we are now dealing with a two-component spinor wave function) $|A|^2$ still measures the incident current, $|B|^2$ the reflected current (and $|C|^2$ the transmitted current), so T retains its essential physical interpretation.

C. Examples

1. Delta functions

Suppose

$$V(x)=g\,\delta(x), \quad A(x)=h\,\delta(x). \tag{156}$$

Integrating Eq. (144) across the singularity gives

$$ic\hbar\alpha\Delta\psi=(g\beta+h)\tilde{\psi}(0), \tag{157}$$

where

$$\tilde{\psi}(0)\equiv\int\delta(x)\psi(x)dx. \tag{158}$$

This integral is notoriously ambiguous,[56] because $\psi(x)$ is discontinuous at the origin. Many authors take $\tilde{\psi}(0)$ to be the *average*:

$$\tilde{\psi}(0)=\tfrac{1}{2}[\psi(0^+)+\psi(0^-)]. \tag{159}$$

This leads to the transfer matrix elements

$$w=1+2\left[\frac{k(g^2-h^2)+2i(Eh+mc^2g)}{k(4\hbar^2c^2-g^2+h^2)}\right], \quad z=4i\frac{(Eg+mc^2h)}{k(4\hbar^2c^2-g^2+h^2)}, \tag{160}$$

which satisfy the constraint $|w|^2-|z|^2=1$, and reduce correctly to Eq. (37) in the nonrelativistic limit (with $g+h$ as the strength of the delta function). But they are inconsistent,[57] as we shall see, with any representation of $\delta(x)$ as the limit of a sequence of finite functions.

2. Rectangular barriers

At a rectangular barrier, with $V(x)=V_0$ and $A(x)=A_0$, $\psi_u(x)$ takes the usual form [Eq. (40)], with [Eq. (147)]

$$k=\frac{1}{\hbar c}\sqrt{E^2-(mc^2)^2}, \quad k'=\frac{1}{\hbar c}\sqrt{(E-A_0)^2-(mc^2+V_0)^2}. \tag{161}$$

The ''lower'' component $\psi_l(x)$ is given by Eq. (146). At the boundaries $(x=\pm a)$, ψ_u and ψ_l are both continuous; this follows from Eq. (145). The transfer matrix is the same as before [Eq. (41)], with k and k' given by Eq. (161), except that in the definition of η,

$$k\to\frac{k}{(E+mc^2)}, \quad k'\to\frac{k'}{(E+mc^2+V_0-A_0)}. \tag{162}$$

In the nonrelativistic regime ($E=mc^2+E_{\rm nr}$, with $E_{\rm nr}$, V_0, and A_0 all $\ll mc^2$) it reduces to the Schrödinger form (Sec. II C 2, with $V_0\to V_0+A_0$). But in the delta-function limit

$$V_0=\frac{g}{2a}, \quad A_0=\frac{h}{2a}, \quad a\to 0, \tag{163}$$

we obtain transfer matrix elements

$$w=\cos\phi-i\alpha_+\sin\phi, \quad z=i\alpha_-\sin\phi, \tag{164}$$

with

$$\phi\equiv\frac{\sqrt{h^2-g^2}}{\hbar c}, \quad \alpha_\pm\equiv\frac{1}{2}\left(\sqrt{\frac{(E-mc^2)(h-g)}{(E+mc^2)(h+g)}}\pm\sqrt{\frac{(E+mc^2)(h+g)}{(E-mc^2)(h-g)}}\right), \tag{165}$$

which is inconsistent with Eq. (160). McKellar and Stephenson[57] show that this result is independent of the shape of the limiting functions used to represent the delta function, and it is clear that the customary prescription [Eq. (159)] cannot be sustained. Equation (160) may describe *some* point interaction,[58] but it is *not* a delta function.

In Fig. 16 we plot the transmission coefficients for locally periodic rectangular barriers.[59] One case is of particular interest: If $V_0=0$, the transmission coefficient for a single barrier remains nonzero as A_0 approaches infinity:

$$T_1\to\frac{1}{1+\left[\frac{mc^2}{E^2-(mc^2)^2}\right]\sin^2\left(\frac{2A_0a}{\hbar c}\right)}. \tag{166}$$

Ordinarily, one would expect the transmission through an infinitely high potential to vanish (as is the case for $V_0\to\infty$). This is a manifestation of the Klein paradox, resulting from pair production at the potential step.[60]

VI. SUMMARY AND CONCLUSION

When a monochromatic wave in one dimension encounters a change in the properties of the propagating medium, reflected and transmitted waves are generated. The transfer matrix **M** relates the incoming and outgoing amplitudes on the left, A and B, to the outgoing and incoming amplitudes on the right, C and D:

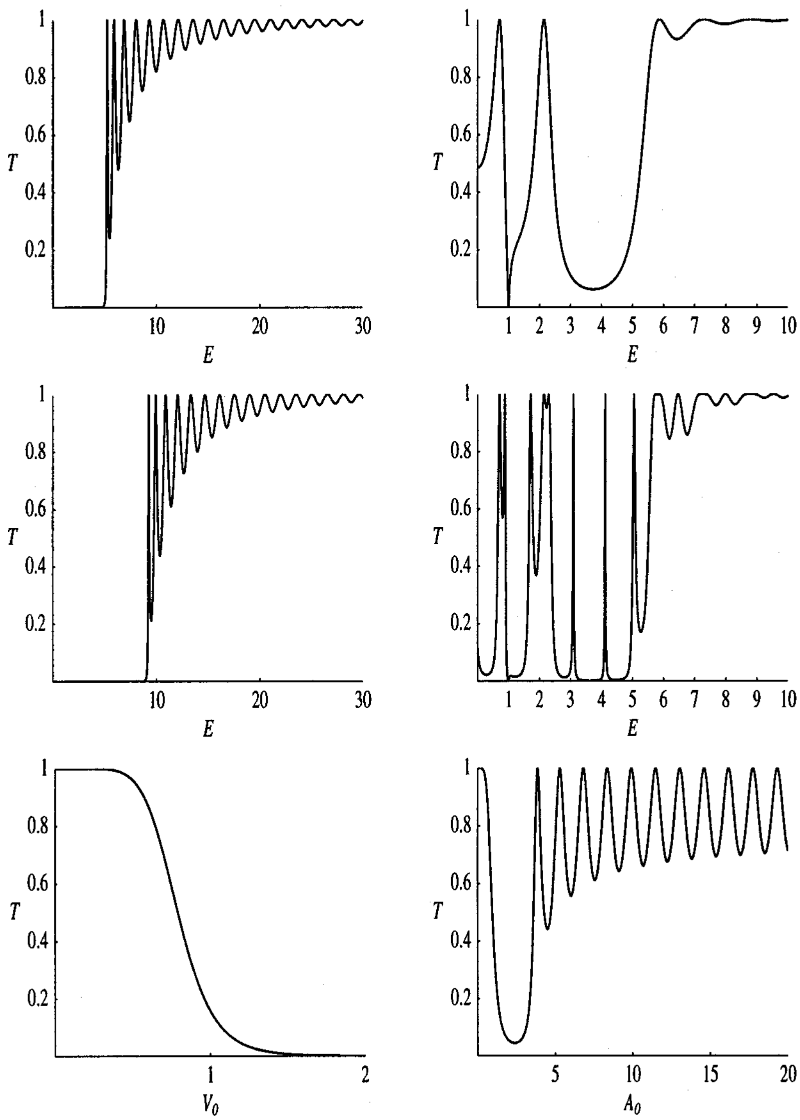

Fig. 16. Transmission coefficients for the Dirac equation with rectangular barriers. In all cases $m=c=\hbar=a=1$, $s=4$. (a) $N=1$, $A_0=0$, $V_0=4$; (b) $N=1$, $A_0=4$, $V_0=0$; (c) $N=1$, $A_0=V_0=4$; (d) $N=2$, $A_0=4$, $V_0=0$; (e) $N=1$, $A_0=0$, $E=2$; (f) $N=1$, $V_0=0$, $E=2$.

$$\begin{pmatrix} A \\ B \end{pmatrix} = \mathbf{M} \begin{pmatrix} C \\ D \end{pmatrix}. \tag{167}$$

Time reversal invariance and the relevant conservation law (energy, mass, charge, or probability) dictate the generic form of the transfer matrix:

$$\mathbf{M} = \begin{pmatrix} w & z \\ z^* & w^* \end{pmatrix}$$

with

$$|w|^2 - |z|^2 = 1. \tag{168}$$

For a locally periodic structure consisting of N repetitions of this ''unit cell,'' a distance s apart, the transfer matrix is

$$\mathbf{M}_N = \begin{pmatrix} [we^{-iks}U_{N-1}(\xi) - U_{N-2}(\xi)]e^{ikNs} & zU_{N-1}(\xi)e^{-ik(N-1)s} \\ z^*U_{N-1}(\xi)e^{ik(N-1)s} & [w^*e^{iks}U_{N-1}(\xi) - U_{N-2}(\xi)]e^{-ikNs} \end{pmatrix}, \tag{169}$$

where k is the wave number, U_N is the Nth Chebychev polynomial of the second kind, and

$$\xi \equiv \tfrac{1}{2}(we^{-iks} + w^*e^{iks}). \tag{170}$$

In particular, the transmission coefficient is

$$T_N = \frac{1}{1 + [|z|U_{N-1}(\xi)]^2}. \tag{171}$$

We have explored a variety of applications, including transverse waves on a weighted string, longitudinal waves on a loaded rod, acoustic waves in a corrugated tube, ocean waves crossing a succession of sandbars, electromagnetic waves in transmission lines, light waves in photonic crystals, and quantum mechanical waves (both nonrelativistic and relativistic) in lattices. This by no means exhausts the possibilities (we have not considered waveguides,[61] for example, or optical fibers with varying index of refraction,[62] or neutron scattering from stratified media,[63] or seismic waves[64]—not to mention chemotherapy[65]—nor have we discussed applications to electrons, photons, and phonons in heterostructures, superlattices, and quantum dots,[66] or—most recently—all-plastic light-emitting diodes[67]). But we have tried to illustrate the *method* in a broad range of contexts. For although the essential result [Eq. (169)] was discovered half a century ago by Abelès,[68] its relevance to the other disciplines has not been widely appreciated.

The characteristic feature of fully periodic systems is band structure: The medium is ''transparent'' in some frequency ranges and ''opaque'' in others. Even with a small number of cells, *locally* periodic systems exhibit precursors of band structure, with intervals of relatively high or low transmission. Equation (171) allows one to explore the emergence of full band structure as N increases.

Perhaps the most remarkable thing about waves in locally periodic media is that the problem can be solved in closed form, for arbitrary N, and the solution can be expressed in a tidy, succinct form. For some applications the fully periodic analysis, as pioneered by Kronig and Penney, is entirely adequate. However, with the increasing sophistication of technology and fabrication, the exact analysis, for a finite number of layers, becomes critical to a complete understanding of the process.

ACKNOWLEDGMENTS

DJG would like to thank Roger Falcone and the Physics Department at the University of California, Berkeley, where the bulk of this work was carried out. I am grateful in particular for many illuminating conversations with Alan Portis. We also thank the anonymous referees for their extraordinarily careful review of the lengthy manuscript.

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[1]R. de L. Kronig and W. G. Penney, ''Quantum Mechanics of Electrons in Crystal Lattices,'' Proc. R. Soc. London, Ser. A **130**, 499–513 (1931).

[2]See, for example, Neil W. Ashcroft and N. David Mermin, *Solid State Physics* (Saunders, Philadelphia, 1976), Chap. 8.

[3]F. Abelès, ''Recherches sur la propagation des ondes électromagnétiques sinusoïdales dans les milieux stratifiés,'' Ann. Phys. (Paris) **5**, 706–782 (1950), especially pp. 777–781.

[4]M. Cvetič and L. Pičman, ''Scattering states for a finite chain in one dimension,'' J. Phys. A **14**, 379–382 (1981). See also P. Erdös and R. C. Herndon, ''Theories of electrons in one-dimensional disordered systems,'' Adv. Phys. **31**, 65–163 (1982). The particular case of a delta function potential was solved earlier by D. Kiang, ''Multiple scattering by a Dirac comb,'' Am. J. Phys. **42**, 785–787 (1974).

[5]**S** and **M** are related, of course: $\mathbf{S}=(1/M_{11})\begin{pmatrix} M_{21} & \det M \\ 1 & -M_{12}\end{pmatrix}$.

[6]E. Merzbacher, *Quantum Mechanics* (Wiley, New York, 1998), 3rd ed., Sec. 6.3.

[7]See Ref. 6, Sec. 3.1.

[8]If the potential is *symmetric*, $V(-x)=V(x)$, one obtains the further condition that z is imaginary. See Ref. 6.

[9]What follows is a variation on the method of D. W. L. Sprung, Hua Wu, and J. Martorell, ''Scattering by a finite periodic potential,'' Am. J. Phys. **61**, 1118–1124 (1993). See also D. W. L. Sprung, J. D. Sigetich, H. Hua, and J. Martorell, ''Bound states of a finite periodic potential,'' Am. J. Phys. **68**, 715–722 (2000).

[10]One method is to diagonalize **P**: $\mathbf{D}^{-1}\mathbf{PD}=\begin{pmatrix} p_+ & 0 \\ 0 & p_-\end{pmatrix}$, where $p_\pm$ are the eigenvalues and **D** is the matrix of eigenvectors. Then $\mathbf{P}^N=\mathbf{D}\begin{pmatrix} p_+^N & 0 \\ 0 & p_-^N\end{pmatrix}\mathbf{D}^{-1}$. Another method uses the Cauchy integral formula for matrices (Kiang, Ref. 4). For details see C. A. Steinke, ''Scattering from a finite periodic potential and the classical analogs,'' senior thesis, Reed College, 1998. For a particularly elegant approach see Hua Wu, D. W. L. Sprung, and J. Martorell, ''Periodic quantum wires and their quasi-one-dimensional nature,'' J. Phys. D **26**, 798–803 (1993).

[11]For a proof see S. Lang's *Linear Algebra* (Springer, New York, 1987), 3rd ed., p. 241.

[12]M. Abromowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1965), pp. 777, 782.

[13]This extraordinary result was apparently first obtained (in the quantum context) by Cvetič and Pičman (Ref. 4), though a quite different analytical solution was given by C. Rorres, ''Transmission Coefficients and Eigenvalues of a Finite One-Dimensional Crystal,'' SIAM (Soc. Ind. Appl. Math.) J. Appl. Math. **27**, 303–321 (1974). It was rediscovered by D. J. Vezzetti and M. M. Cahay, ''Transmission resonances in finite, repeated structures,'' J. Phys. D **19**, L53–L55 (1986); again by T. M. Kalotas and A. R. Lee, ''One-dimensional quantum interference,'' Eur. J. Phys. **12**, 275–282 (1991), by Sprung, Wu, and Martorell (Ref. 9), and yet again by M. G. Rozman, P. Reineker, and R. Tehver, ''Scattering by locally periodic one-dimensional potentials,'' Phys. Lett. A **187**, 127–131 (1994). See also N. L. Chuprikov, ''Tunneling in a one-dimensional system of N identical potential barriers,'' Semiconductors **30**, 246–251 (1996); R. Pérez-Alvarez and H. Rodriguez-Coppola, ''Transfer Matrix in 1D Schrödinger Problems with Constant and Position-Dependent Mass,'' Phys. Status Solidi B **145**, 493–500 (1988); E. Liviotti, ''Transmission through one-dimensional periodic media,'' Helv. Phys. Acta **67**, 767–768 (1994); and P. Erdös, E. Liviotti, and R. C. Herndon, ''Wave transmission through lattices, superlattices and layered media,'' J. Phys. D **30**, 338–345 (1997). These authors used a variety of different notations, and not all of them realized that the polynomials in question were Chebychev's. Others found the transmission coefficient for particular potentials, but did not realize that the result generalizes: Kiang (Ref. 4), D. J. Griffiths and N. F. Taussig, ''Scattering from a locally periodic potential,'' Am. J. Phys. **60**, 883–888 (1992). Asymptotic forms, and transmission times, were considered by F. Barra and P. Gaspard, ''Scattering in periodic systems: From resonances to band structure,'' J. Phys. A **32**, 3357–3375 (1999). For generalizations to three-dimensional and multichannel systems, see, for example, P. Pereyra, ''Non-commutative polynomials and the transport properties in multichannel-multilayer systems,'' *ibid.* **31**, 4521–4531 (1998). For a radically different approach to the whole problem see M. G. E. da Luz, E. J. Heller, and B. K. Cheng, ''Exact form of Green functions for segmented potentials,'' *ibid.* **31**, 2975–2990 (1998).

[14]It is related to the eigenfunctions of **P**: $p_\pm=\exp(\pm i\gamma)$. In the limit $N\rightarrow\infty$ there will be total reflection when ξ is outside of the range $(-1,+1)$.

[15]Griffiths and Taussig, Ref. 13. Incidentally, the most general point interaction satisfies the boundary conditions $\psi(0^+)+\bar{\gamma}\psi(0^-)=-\bar{\delta}\psi'(0^-)$; $\psi'(0^+)+\bar{\alpha}\psi'(0^-)=-\bar{\beta}\psi(0^-)$, where $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$, and $\bar{\delta}$ are real parameters subject only to the constraint $\bar{\alpha}\bar{\gamma}-\bar{\beta}\bar{\delta}=1$. See F. A. B. Coutinho, Y. Nogami, and J. F. Perez, ''Generalized point interactions in one-dimensional quantum mechanics,'' J. Phys. A **30**, 3937–3945 (1997). The delta-function potential [Eq. (36)] is the special case $\bar{\alpha}=\bar{\gamma}=-1$, $\bar{\delta}=0$, $\bar{\beta}=-c$. For the general case, $w=-\frac{1}{2}[(\bar{\alpha}+\bar{\gamma})+i(\bar{\beta}/k-\bar{\delta}k)]$, $z=-\frac{1}{2}[(\bar{\alpha}-\bar{\gamma})+i(\bar{\beta}/k+\bar{\delta}k)]$.

[16]Kiang, Ref. 4; D. Lessie and J. Spadaro, ''One-dimensional multiple scattering in quantum mechanics,'' Am. J. Phys. **54**, 909–913 (1986); H.-W. Lee, A. Zysnarski, and P. Kerr, ''One-dimensional scattering by a locally periodic potential,'' *ibid.* **57**, 729–734 (1989); Kalotas and Lee (Ref. 13); Griffiths and Taussig (Ref. 13); Sprung, Wu, and Martorell (Ref. 9).

[17]The delta-function example is explored in Griffiths and Taussig (Ref. 13); delta functions and rectangular barriers are treated in P. Carpena, V. Gasparian, and M. Ortuño, ''Number of bound states of a Kronig–Penney finite-periodic superlattice,'' Euro. Phys. J. B **8**, 635–641 (1999); for the general case see M. Sassoli de Bianchi and M. Di Ventra, ''On the number of states bound by one-dimensional finite periodic potentials,'' J. Math. Phys. **36**, 1753–1764 (1995); D. W. L. Sprung, Hua Wu, and J. Martorell,

''Addendum to 'Periodic quantum wires and their quasi-one-dimensional nature,' '' J. Phys. D **32**, 2136–2139 (1999).

[18]This case, and also the finite well, were explored numerically by E. Cota, J. Flores, and G. Monsivais, ''A simple way to understand the origin of the electron band structure,'' Am. J. Phys. **56**, 366–372 (1988). It was studied analytically by Gregory Elliott (personal communication).

[19]A program called ''ENERGY BAND CREATOR'' (with a graphical interface) has been written by a group at Kansas State University to calculate these energies for up to 50 cells. It is available at http://www.phys.ksu.edu/perg/vqm/programs.

[20]For a subtle and illuminating perspective on some of these problems see H. Georgi, *The Physics of Waves* (Prentice–Hall, Englewood Cliffs, NJ, 1993).

[21]The fully periodic case ($N \rightarrow \infty$) was studied by U. Oseguera, ''Classical Kronig–Penney model,'' Am. J. Phys. **60**, 127–130 (1992). For interesting historical commentary see I. B. Crandall, *Theory of Vibrating Systems and Sound* (van Nostrand, New York, 1926), Sec. 26; L. Brillouin, *Wave Propagation in Periodic Structures* (McGraw–Hill, New York, 1946).

[22]The minus sign indicates that this system is analogous to the delta-function *well* (negative c), and one might wonder whether there exist classical analogs to quantum bound states—standing waves in the weighted zone, with exponential attenuation outside. But this would require an imaginary k, which is possible in the quantum case ($k=\sqrt{2mE}/\hbar$), when $E<0$, but not in the classical one ($k=\omega\sqrt{\mu/T}$), unless we are prepared to countenance strings with negative mass or negative tension. (Actually, the latter *is* realizable, if we use a long stiff watch spring under compression.)

[23]This makes a nice demonstration—we used fishing weights of about half a gram, and measured the normal mode frequencies for various N (Steinke, Ref. 10). For a numerical and experimental study, see S. Parmley *et al.*, ''Vibrational properties of a loaded string,'' Am. J. Phys. **63**, 547–553 (1995).

[24]See, for example, W. C. Elmore and M. A. Heald, *Physics of Waves* (Dover, New York, 1969), Sec. 4.1.

[25]In the case of varying S, we shall assume that the change is gradual enough that the force remains uniformly distributed over the cross section.

[26]See, for example, S. Temkin, *Elements of Acoustics* (Wiley, New York, 1981), Chap. 2.

[27]There are related applications to underwater acoustics, but these involve a more complicated fluid, and *nonperiodic* variations. L. M. Brekhovskikh, *Waves in Layered Media* (Academic, Orlando, FL, 1980), 2nd ed.

[28]Temkin, Ref. 24, Sec. 2.3.

[29]Brekhovskikh, Ref. 25, Chap. II; A. Alippi, A. Bettucci, and F. Craciun, ''Ultrasonic waves in monodimensional periodic composites,'' in *Physical Acoustics: Fundamentals and Applications*, edited by O. Leroy and M. A. Breazeale (Plenum, New York, 1990). For the classic papers on acoustic filters see R. B. Lindsay, *Physical Acoustics* (Dowden, Hutchinson, and Ross, Stroudsburg, PA, 1974), Secs. 6–8.

[30]Transverse acoustic modes can be suppressed by maintaining the frequency below the ''cutoff'' $\sim v/\sqrt{S}$. See Temkin, Ref. 26, Chap. 3. For an interesting experimental approach, which could easily be adapted to the systems discussed here, see C. L. Adler, K. Mita, and D. Phipps, ''Quantitative measurement of acoustic whistlers,'' Am. J. Phys. **66**, 607–612 (1998).

[31]The horn equation is traditionally attributed to Webster, who published it in 1919, but it was in fact studied by many others, from Daniel Bernoulli to Lord Rayleigh. For a fascinating account, with extensive bibliography, see E. Eisner, ''Complete Solutions of the 'Webster' Horn Equation,'' J. Acoust. Soc. Am. **41**, 1126–1146 (1967).

[32]Temkin, Ref. 26, treats the exponential and power-law profiles in Sec. 3.8; for a complete listing see Eisner, Ref. 31, p. 1128.

[33]Temkin, Ref. 26, Sec. 3.9.

[34]See, for instance, L. E. Kinsler, A. R. Frey, A. B. Coppens, and J. V. Sanders, *Fundamentals of Acoustics* (Wiley, New York, 1982), pp. 231–243. For interesting related work see J. V. Sánchez-Pérez *et al.*, ''Sound Attenuation by a Two-Dimensional Array of Rigid Cylinders,'' Phys. Rev. Lett. **80**, 5325–5328 (1998).

[35]F. S. Crawford, ''Singing corrugated pipes,'' Am. J. Phys. **42**, 278–288 (1974). This article is a real gem.

[36]If there is no corrugation at all ($S_1=S_2$), then $\varepsilon_+=1$, $\varepsilon_-=0$, and $\xi=\cos(4ka)$, so $\gamma=4ka$ [Eq. (34)], and Eq. (112) reduces to $\sin[(N+1)ks]=0$. But $(N+1)s=L$, the length of the tube, and (happily) we recover the familiar standing wave formula $\lambda_n=2L/n$.

[37]Crawford (Ref. 35) takes the corrugations to be sinusoidal, but a comparison of Figs. 11 and 13 tends to confirm one's intuition that the details of the profile are not terribly critical.

[38]Crawford's measured value was 175 Hz; presumably the end corrections are about the same for smooth and corrugated pipes. Incidentally, the corrugahorn is ill-tempered (the overtones are not perfect harmonics), which perhaps accounts for the fact that it has always been more popular with physicists than musicians—though according to our figures its bad temper is rather mild.

[39]See, for example, M. Rahman, *Water Waves: Relating Modern Theory to Advanced Engineering Practice* (Oxford U.P., Oxford, 1995), Chap. 4.

[40]Deep water waves are not so interesting, from our present perspective, since v is independent of the depth, and there is no realistic way to provide for local periodicity.

[41]C. C. Mei, *The Applied Dynamics of Ocean Surface Waves* (Wiley, New York, 1983), Chap. 4.

[42]A set of artificial shoals could conceivably be constructed outside a harbor to exclude waves in a particularly destructive frequency range, but as far as we know this has not been tried. Nor do we know of any naturally occurring examples.

[43]Physical oceanographers typically invoke the Wentzel–Kramers–Brillouin approximation to handle Eq. (120). See Mei, Ref. 41, Sec. 4.5, or M. W. Dingemans, *Water Wave Propagation over Uneven Bottoms* (World Scientific, Singapore, 1997), Sec. 2.6.

[44]For shallow waves the horizontal velocity is effectively independent of the vertical coordinate. See A. Defant, *Physical Oceanography* (Pergamon, New York, 1961), Vol. 2, p. 142.

[45]For an amusing historical commentary see W. Bascom, *Waves and Beaches: The Dynamics of the Ocean Surface* (Anchor, Garden City, NY, 1980), p. 142.

[46]Lord Kelvin analyzed the case of water *flowing* down a channel with sinusoidal depth [W. Thomson, ''On Stationary Waves in Flowing Water,'' Philos. Mag. **22**, 353–357 (1886)]; see also D. E. Hewgill, J. Reeder, and M. Shinbrot, ''Some Exact Solutions of the Nonlinear Problem of Water Waves,'' Pac. J. Math. **92**, 87–109 (1981). This system, which corresponds to the *played* corrugahorn, was studied experimentally by T. Shinbrot, ''On Salient Phenomena of Stationary Waves in Water,'' senior thesis, Reed College, 1978. But only modes with wavelengths comparable to the corrugation distance were explored, and it would be interesting to see whether other modes could be stimulated by Crawford's mechanism. B. J. Korgen [''Seiches,'' Am. Sci. **83**, 331–341 (1995)] discusses a related phenomenon, in which standing waves are generated when the upper layer of ocean water flows over a sequence of internal solitons at the thermocline level.

[47]R. B. Prigo, T. O. Manley, and B. S. H. Connell, ''Linear, one-dimensional models of the surface and internal standing waves for a long and narrow lake,'' Am. J. Phys. **64**, 288–300 (1996).

[48]For an introduction to the theory of transmission lines, see (for instance) N. N. Rao, *Elements of Engineering Electromagnetics* (Prentice Hall, Upper Saddle River, NJ, 1991), 3rd ed., Sec. 7.1.

[49]If the space between the conductors is filled with linear insulating material of permittivity ϵ and permeability μ, then $\mathcal{LC}=\epsilon\mu$, and $v=1/\sqrt{\epsilon\mu}$. For an air-filled transmission line $v=1/\sqrt{\epsilon_0\mu_0}=c$, the speed of light.

[50]See, for example, D. J. Griffiths, *Introduction to Electrodynamics* (Prentice Hall, Upper Saddle River, NJ, 1999), 3rd ed., Sec. 9.3.

[51]The transfer matrix for this problem (including oblique incidence) was obtained by Abelès (Ref. 3) in 1950. The most accessible reference in English is M. Born and E. Wolf, *Principles of Optics* (Pergamon, New York, 1980), 6th ed., Sec. 1.6.5. See also P. Yeh, A. Yariv, and C.-S. Hong, ''Electromagnetic propagation in periodic stratified media. I. General theory,'' J. Opt. Soc. Am. **67**, 423–438 (1977); U. Bandelow and U. Leonhardt, ''Light propagation in one-dimensional lossless dielectrica: Transfer matrix method and coupled mode theory,'' Opt. Commun. **101**, 92–99 (1993); J. Lekner, ''Light in periodically stratified media,'' J. Opt. Soc. Am. A **11**, 2892–2899 (1994); J. M. Bendickson, J. P. Dowling, and M. Scalora, ''Analytic expressions for the electromagnetic mode density in finite, one-dimensional, photonic band-gap structures,'' Phys. Rev. E **53**, 4107–4121 (1996); Vadim B. Kazanskiy and Vladimir V. Podlozny, ''Resonance Phenomena in Finite-Periodic Multilayer Sequence of Identical Gratings of Rectangular Bars,'' Microwave Opt. Technol. Lett. **21**, 299–304 (1999).

[52]Layered optical media have found important applications as lens coatings, x-ray mirrors, and photonic crystals. See A. G. Michette, *Optical Systems for Soft X-Rays* (Plenum, New York, 1986); P. Boher and P. Houdy, ''Multicouches nanométriques pour l'optique X: Quelques exemples ap-

plicables aux lasers X,'' Ann. Phys. (Paris) **17**, 141–150 (1992); E. Spiller, *Soft X-Ray Optics* (SPIE, Bellingham, WA, 1994), Chap. 7; E. Yablonovitch, ''Inhibited Spontaneous Emission in Solid-State Physics and Electronics,'' Phys. Rev. Lett. **58**, 2059–2062 (1987); J. D. Joannopoulos, R. D. Meade, and J. N. Winn, *Photonic Crystals: Molding the Flow of Light* (Princeton U.P., Princeton, NJ, 1995).

[53]See, for example, C. Itzykson and J.-B. Zuber, *Quantum Field Theory* (McGraw–Hill, New York, 1980), Sec. 2-1-2.

[54]From now on Ψ will always stand for the *two*-component spinor [Eq. (141)]. There are other representations for the gamma matrices, and they lead to different expressions for α and β; but they are all equivalent, corresponding to different choices for the basis spinors.

[55]In the relativistic context, E includes the rest energy: $E = E_{\mathrm{nr}} + mc^2$. Thus $k = \sqrt{E_{\mathrm{nr}}^2 + 2E_{\mathrm{nr}}mc^2}/\hbar c$, and in the nonrelativistic limit ($E_{\mathrm{nr}} \ll mc^2$), $k \approx \sqrt{2mE_{\mathrm{nr}}}/\hbar$, consistent with Eq. (4).

[56]See D. J. Griffiths and S. Walborn, ''Dirac deltas and discontinuous functions,'' Am. J. Phys. **67**, 446–447 (1999), and references therein.

[57]M. G. Calkin and D. Kiang, ''Proper treatment of the delta function potential in the one-dimensional Dirac equation,'' Am. J. Phys. **55**, 737–739 (1987); B. H. J. McKellar and G. J. Stephenson, Jr., ''Relativistic quarks in one-dimensional periodic structures,'' Phys. Rev. C **35**, 2262–2271 (1987).

[58]See Ref. 15.

[59]For related work see M. L. Glasser, ''A class of one-dimensional relativistic band models,'' Am. J. Phys. **51**, 936–939 (1983), and references cited therein.

[60]McKellar and Stephenson, Ref. 57. See also Barry R. Holstein, ''Klein's paradox,'' Am. J. Phys. **66**, 507–512 (1968).

[61]J. O. Vasseur *et al.*, ''Absolute band gaps and electromagnetic transmission in quasi-one-dimensional comb structures,'' Phys. Rev. B **55**, 10434–10442 (1997).

[62]N. Kashima, *Passive Optical Components for Optical Fiber Transmission* (Artech House, Boston, MA, 1995), Chap. 11.

[63]J. Lekner, ''Reflection of neutrons by periodic stratifications,'' Physica B **202**, 16–22 (1994).

[64]F. Meseguer *et al.*, ''Raleigh-wave attenuation by a semi-infinite two-dimensional elastic-band-gap crystal,'' Phys. Rev. B **59**, 12169–12172 (1999).

[65]J. Adam and J. C. Panetta, ''A Simple Mathematical Model and Alternative Paradigm for Certain Chemotherapeutic Regimens,'' Math. Comput. Modelling **22**, 49–60 (1995).

[66]R. Tsu and L. Esaki, ''Tunneling in a finite superlattice,'' Appl. Phys. Lett. **22**, 562–564 (1973); R. M. Kolbas and N. Holonyak, Jr., ''Man-made quantum wells: A new perspective on the finite square-well problem,'' Am. J. Phys. **52**, 431–437 (1984); L. P. Kouwenhoven *et al.*, ''Transport through a Finite One-Dimensional Crystal,'' Phys. Rev. Lett. **65**, 361–364 (1990); S. Mizuno and S. Tamura, ''Transmission of Sub-THz Acoustic Phonons in a Kronig–Penny (sic.) System for Phonons,'' Solid State Commun. **85**, 639–642 (1993). See also M. O. Vassell, J. Lee, and H. F. Lockwood, ''Multibarrier tunneling in $\mathrm{Ga}_{1-x}\mathrm{Al}_x\mathrm{As/GaAs}$ heterostructures,'' J. Appl. Phys. **54**, 5206–5213 (1983); P. Yeh, ''Resonant tunneling of electromagnetic radiation in superlattice structures,'' J. Opt. Soc. Am. A **2**, 568–571 (1985); H. Wu *et al.*, ''Quantum wire with periodic serial structure,'' Phys. Rev. B **44**, 6351–6360 (1991); P. S. Deo and A. M. Jayannavar, ''Quantum waveguide transport in serial stub and loop structures,'' Phys. Rev. B **50**, 11629–11639 (1994).

[67]P. K. H. Ho *et al.*, ''All-Polymer Optoelectronic Devices,'' Science **285**, 233–236 (1999).

[68]Abelès may not have realized that his method applies to *arbitrary* variations within each cell. Earlier work of W. Shockley [''On the Surface States Associated with a Periodic Potential,'' Phys. Rev. **56**, 317–323 (1939)] treats the general case (but only for specific small *N*).

OCKHAM'S RAZOR

Science treads everywhere, and worms itself under the scabs that religion regards as protecting the special tender patches of human existence. The religious go to intellectual war to maintain that in some areas secularly inspired logic cannot tread. Yet reductionist science is omnicompetent. Science has never encountered a barrier that it has not surmounted or that we can at least reasonably suppose it has power to surmount and will in due course be equipped to do so. There is no explicitly demonstrated validity in the view that there are aspects of the universe closed to science. I can accept, given the success with which science has encroached on the territory once regarded as traditionally religion's, that many people hope its domain of competence will prove bounded, with things of the spirit on that side of the fence and things of the flesh on this. But until it is proved otherwise, there is no reason to suppose that science is incompetent when it brings its razor to bear on belief.

P. W. Atkins, ''The Limitless Power of Science,'' in *Nature's Imagination—The Frontiers of Scientific Vision*, edited by John Cornwell (Oxford University Press, New York, 1995).