

Kater's Pendulum

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Abstract

In this lab we will determine the value of the acceleration of gravity g by using a reversible pendulum, first developed by Henry Kater in 1815.

1 History

During the 18th and 19th centuries the field of geodesy became a quantitative science. The goal of scientists was to determine the shape of the earth as well as how the strength of gravity changed from location to location. The determination of g at these different locations helped to give scientists a better understanding of what the earth was made of. This information was particularly helpful in mining operations. In early experiments, g was determined using the familiar equation for the period of a simple pendulum,

$$T = 2\pi\sqrt{\frac{L}{g}}. \quad (1)$$

This method is of limited utility in making precision measurements of g , as Eq. 1 strictly applies only to a true simple pendulum, consisting of a point mass swinging on a massless rod or string. For a real, or physical, pendulum, the period is given as

$$T = 2\pi\sqrt{\frac{I}{mgl}}. \quad (2)$$

Here, I is the moment of inertia of the pendulum about the pivot point, m is the pendulum's mass, and l is the distance from this pivot to the center of mass. Even for a carefully constructed physical pendulum the quantities I and l are difficult to measure with great precision.

In 1815, Henry Kater developed a *reversible* pendulum, with which he could make measurements of g to unprecedented accuracy. Kater's key breakthrough was to use a pendulum that could be swung from either of two pivots, located at opposite ends of the pendulum. The periods when the pendulum is swung from each of the pivots are recorded. When the mass distribution of the pendulum is adjusted so that these two periods were *equal*, the difficult-to-measure quantities I and l cancel out, and Eq. 1 applies again exactly.



Figure 1: Henry Kater.



Figure 2: Kater's original reversible pendulum.

2 Theory

2.1 The period of a physical pendulum

Consider a physical pendulum that can be swung from either of two pivots. To make the argument concrete, our discussion will be in terms of the pendulum used in this lab. The pendulum has two sets of “knife edge” pivots, parallel to each other, from which the pendulum can swing. Because the knife edges are sharp and parallel, the distance D between them can be precisely measured. This is a key aspect of the reversible pendulum.

Defining the *radius of gyration* k as

$$k = \sqrt{\frac{I}{m}}, \quad (3)$$

Eq. 2 for the period of a physical pendulum can be written as

$$T = 2\pi\sqrt{\frac{k^2}{gl}}. \quad (4)$$

From the parallel-axis theorem we have

$$k^2 = k_{\text{cm}}^2 + l^2, \quad (5)$$

where k_{cm} is the radius of gyration about the center of mass and, as before, l is the distance from the center of mass to the pivot—i.e., to one of the knife edges. Then Eq. 4 becomes

$$T = 2\pi\sqrt{\frac{k_{\text{cm}}^2 + l^2}{gl}}. \quad (6)$$

Let T_1 and T_2 be the periods when the pendulum is swung from pivots 1 and 2 respectively; l_1 and l_2 are the distances from pivots 1 and 2 to the center of mass. Then we can write

$$T_1 = 2\pi\sqrt{\frac{k_{\text{cm}}^2 + l_1^2}{gl_1}} \quad \text{and} \quad T_2 = 2\pi\sqrt{\frac{k_{\text{cm}}^2 + l_2^2}{gl_2}} \quad (7)$$

Solving for k_{cm}^2 , the fixed radius of gyration about the center of mass, we find that

$$k_{\text{cm}}^2 = \frac{gl_1 T_1^2}{4\pi^2} - l_1^2 = \frac{gl_2 T_2^2}{4\pi^2} - l_2^2, \quad (8)$$

or

$$l_1 T_1^2 - l_2 T_2^2 = \frac{4\pi^2}{g} (l_1^2 - l_2^2) \quad (9)$$

After some manipulation, Eq. 9 gives

$$\frac{8\pi^2}{g} = \frac{T_1^2 + T_2^2}{l_1 + l_2} + \frac{T_1^2 - T_2^2}{l_1 - l_2}. \quad (10)$$

Note that $l_1 + l_2 = D = 40.913 \pm 0.003$ cm, the distance between the knife edges. Eq. 10 gives g in terms of the two periods and the two distances l . The right-hand side of the equation can be viewed as an overall value $(T_1^2 + T_2^2)/D$ plus a correction term. This correction term vanishes as T_1 approaches T_2 . In the case where $T_1 = T_2 \equiv T$ exactly we have the simple result

$$g = \frac{4\pi^2 D}{T^2} \quad (11)$$

2.2 Amplitude variation of the period

The results derived above are based on solving the differential equation for the physical pendulum in the limit of small amplitude oscillations. When the amplitude becomes large, we will need a more precise result. Using conservation of energy we can write

$$mgl(\cos \theta - \cos \theta_0) = \frac{1}{2} I \dot{\theta}^2,$$

where θ_0 is the maximum angle the pendulum reaches. We can rewrite this as

$$dt = \sqrt{\frac{I}{2mgl}} \frac{d\theta}{\sqrt{\cos \theta - \cos \theta_0}} = \frac{T_0 d\theta}{2\pi\sqrt{2}\sqrt{\cos \theta - \cos \theta_0}},$$

where we have used the fact that the period T_0 for infinitesimal oscillations is given by Eq. 2 as $T_0 = 2\pi(I/mgl)^{1/2}$. Integrating both sides over a quarter period gives

$$\int_0^{T/4} dt = \frac{T}{4} = \int_0^{\theta_0} \frac{T_0 d\theta}{2\pi\sqrt{2}\sqrt{\cos \theta - \cos \theta_0}}, \quad (12)$$

or

$$T = \frac{\sqrt{2}T_0}{\pi} \int_0^{\theta_0} \frac{d\theta}{\sqrt{\cos \theta - \cos \theta_0}} \quad (13)$$

The integral in Eq. 13 can be evaluated in terms of elliptic integrals. However, we can derive an expansion for the period that will be accurate enough for our purposes. Using the identity $\cos x = 1 - 2\sin^2(x/2)$, we can write Eq. 13 as

$$T = \frac{T_0}{\pi} \int_0^{\theta_0} \frac{d\theta}{\sqrt{\sin^2(\theta_0/2) - \sin^2(\theta/2)}} \quad (14)$$

We now make the substitution $\sin(\theta/2) = \sin(\theta_0/2) \sin \phi$, giving, after some algebra, simply

$$T = \frac{2T_0}{\pi} \int_0^{\pi/2} \frac{d\phi}{\cos(\theta/2)} \quad (15)$$

or

$$T = \frac{2T_0}{\pi} \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - \sin^2(\theta_0/2) \sin^2 \phi}} \quad (16)$$

Because θ_0 is small, we can expand the integrand in powers of $\sin \theta_0$. To lowest non-vanishing order, this gives

$$T = \frac{2T_0}{\pi} \int_0^{\pi/2} \left(1 + \frac{1}{2} \sin^2 \frac{\theta_0}{2} \sin^2 \phi + \dots \right) d\phi \quad (17)$$

$$= T_0 \left(1 + \frac{1}{\pi} \sin^2 \frac{\theta_0}{2} \int_0^{\pi/2} \sin^2 \phi d\phi \right) \quad (18)$$

$$= T_0 \left(1 + \frac{1}{4} \sin^2 \frac{\theta_0}{2} \right) \approx T_0 \left(1 + \frac{\theta_0^2}{16} \right). \quad (19)$$

Thus the finite-amplitude period is always *longer* than the zero-amplitude period T_0 .

3 Procedure

The basic goal of this lab is to make a precision measurement of g using a Kater's pendulum. By adjusting the two weights in a systematic way, you should be able to get the two periods T_1 and T_2 to be quite close. Then Eq. 10 can be used to solve for g .

Although the basic idea of the lab is simple, there are many issues that you will need to think about as you try to get your best measurement of g . Here are a few things you might want to think about, in no particular order:

- How will you handle the amplitude variation of the period? You can try to use amplitudes small enough that the variation is negligible, but you might find that the detector is very noisy in this regime, because the pendulum is moving so slowly. A plot of T vs. θ_0 would be interesting to compare with Eq. 19.
- What errors are introduced simply by setting the pendulum down on the bracket?
- Do air currents have an appreciable effect? How about people walking about?
- You can't ever get the two periods exactly equal. But you can think of the rightmost term in Eq. 10 as a correction term that goes to zero as T_1 approaches T_2 . How small is it? How would you measure l_1 and l_2 , and how accurately would they need to be measured?
- Where should the photogate be placed? Is there any effect on the period as you try different locations for it?

These days, g is measured by directly by timing an object in free fall. Although this seems like the most obvious way to measure g , this method is actually quite tricky because the object falls at a modestly high speed, and because its position needs to be measured very accurately while its falling. By using a laser interferometer to measure the object's position, however, and by performing the measurement in a high vacuum to negate the effects of air resistance, such instruments can now routinely measure g to about 1 part in a billion!

Although g has not been measured at our site with such an instrument, discussions with scientists at the National Geodetic Survey show that the value in our lab is $9.79726 \pm 0.00002 \text{ m/s}^2$.

4 Further Refinements

There are several other corrections that must be made in order to calculate g to the highest possible accuracy [1]. First, the buoyancy of the pendulum in air slightly decreases the gravitational torque on the pendulum compared to its value in a vacuum. This tends to increase the pendulum's period. Second, as the pendulum swings it effectively carries with it a certain volume of air that surrounds it. This "added-mass" correction is small but not negligible. It is also quite difficult to calculate [2]. See Refs. [1] and [2] for more information.

References

- [1] D. Candela, K. M. Martini, R. V. Krotkov, and K. H. Langley, *Bessel's improved Kater pendulum in the teaching lab*, Am. J. Phys. **69**, 714 (2001).
- [2] R. A. Neslon and M. G. Olsson, *The pendulum—Rich physics from a simple system*, Am. J. Phys. **54**, 112 (1986).