Demonstration for observing $J_0(x)$ on a resonant rotating vertical chain

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A simple example of mechanical resonance using a light chain rotating about a vertical axis is described. The equation of motion for the chain can be reduced to Bessel's function of the first kind of order zero [$J_0(x)$]. Stable lobe patterns are formed with eigenfrequencies and internode spacing which agree well with the zeros of $J_0(x)$.

INTRODUCTION

Upper division physics and engineering students have numerous occasions to study physical systems in resonance. Often times such resonances involve the solutions to eigenvalue problems giving rise to special functions. Unfortunately there are few such examples which bridge the gap between simple standing waves on a stretched string and the involved calculations of complicated mechanical systems or the quantum mechanics of the hydrogen atom. Fewer still are simple problems involving special functions that can be demonstrated easily in the classroom.

One relatively simple example which introduces Bessel's function is a flexible chain supported at one end in a uniform gravitational field and oscillating in a vertical plane. Slattery has discussed this problem, and several other examples of mechanical oscillation.[1] More recently other authors have expanded on this theme.[2,3] Unfortunately, as a demonstration or laboratory exercise, the swaying chain is limited since, to quote Slattery, "It is quite difficult to make the chain vibrate in its first overtone and many attempts may be necessary to get a good shape with a stationary node." This shortcoming is overcome for a chain rotating about a vertical axis through the point of suspension.

A chain hanging freely by one end from the center of a rotating rod exhibits characteristic resonances similar to standing waves on a string, as shown in Fig. 1. The wave form, however, approximates that of the zero-order Bessel function, $J_0(k\sqrt{z})$, where z is the vertical distance from the free end. A demonstration of this effect can be assembled in extremely short time using a necklace chain and a variable speed rotor. Measurements of the internode distances of a chain in resonance, when compared to the zeros of J_0, predict resonance frequencies in good agreement with the measured ones. Best of all, students can actually observe a Bessel function right before their very eyes!

DERIVATION OF BESSEL'S EQUATION

The mathematics is straight forward. One assumes small deflections and small slopes in a flexible chain of linear mass density, λ, and length L. Referring to Fig. 2, let r be the horizontal distance from the z axis to a point on the chain, with $z = L$ the point of suspension. One seeks stable configurations of the form $r = f(z)$.[4] The net horizontal force, $F_{r\,\mathrm{net}}$, on an element Δm of the chain is simply

$$F_{r\,\mathrm{net}} = T_{\mathrm{hor}} - (T_{\mathrm{hor}} + \Delta T_{\mathrm{hor}}) = -\Delta T_{\mathrm{hor}}, \qquad (1)$$

where T_{hor} is the horizontal component of the tension. This component of the tension is given by $T_{\mathrm{hor}} = T\sin\beta$, where β is the angle between the chain and the vertical. For small β, $\sin\beta \doteq \tan\beta$, and the net horizontal force on a small mass element Δm is

$$F_{r\,\mathrm{net}} = -\Delta(T\tan\beta) = -\Delta\left(T\frac{dr}{dz}\right), \qquad (2)$$

for small slopes, an element of arc length Δs is approximately Δz and thus $\Delta m \doteq \lambda\Delta z$. Newton's second law then gives

$$-\Delta\left(T\frac{dr}{dz}\right) = (\lambda\Delta z)\,(r\omega^2), \qquad (3)$$

where ω is the angular frequency of the rotation. If the centripetal acceleration $r\omega^2$ is small compared to the acceleration of gravity g, the tension is approximately unaffected by the rotation. Then $T = \lambda g z$ and upon dividing through by Δz and taking the limit as Δz goes to zero, one obtains

$$\frac{d}{dz}\left(\lambda g z\frac{dr}{dz}\right) = -r\omega^2 \qquad (4)$$

or

$$z\frac{d^2r}{dz^2} + \frac{dr}{dz} + \frac{r\omega^2}{g} = 0. \qquad (5)$$

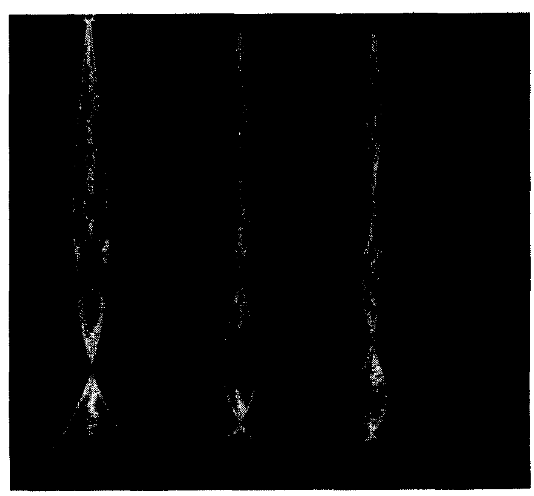

Fig. 1. Time exposure photographs show stable resonances of a rotating flexible chain.

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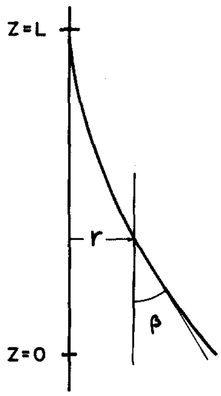

Fig. 2. One seeks solutions of the form $r = f(z)$. At the point of suspension, $z = L$, one requires $r = 0$.

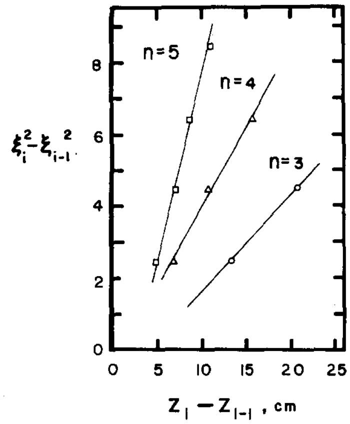

Fig. 3. The difference between the squares of successive zeros of the Bessel function J_0 are plotted against internode distances for three different stable resonances containing 3, 4, and 5 nodes.

Upon substituting $x = 2\omega(z/g)^{1/2}$, Eq. (5) becomes

$$x^2 \frac{d^2r}{dx^2} + x\frac{dr}{dx} + x^2 r = 0, \tag{6}$$

which is Bessel's equation of order zero. The general solution is

$$r = A\,J_0(x) + B\,Y_0(x). \tag{7}$$

We dismiss the $Y_0(x)$ solutions, requiring finite solutions at the chain end. Thus one has stable time-independent solutions of the form

$$r = J_0[2\omega(z/g)^{1/2}]. \tag{8}$$

Since the point of suspension is a node ($r = 0$), the eigenfrequencies, ω_n, for which the chain resonates occur when $2\omega_n(L/g)^{1/2} = \xi_n$, the nth zero of J_0.

In practice $J_0(0) = 1$, thus the lowest segment of the chain has large amplitude violating the assumptions used to derive the above expression for the eigenfrequencies. If, however, one ignores this lowest segment and considers only that portion of the chain between the lowest node and the point of suspension, good agreement with the predicted eigenfrequencies is obtained. In particular one writes

$$\xi_i = 2\omega_n(z_i/g)^{1/2}, \tag{9}$$

where n equals the number of nodes present and $i = 2, 3, \ldots n$. Upon squaring and taking the difference between adjacent nodes one obtains

$$\xi_i^2 - \xi_{i-1}^2 = (4\,\omega_n^2/g)(z_i - z_{i-1}). \tag{10}$$

The left-hand side of Eq. (10) is easily calculated using a table of the zeros of the Bessel function J_0,[5] while the $(z_i - z_{i-1})$ are just the internode distances on the resonant chain. Thus if one plots $(\xi_i^2 - \xi_{i-1}^2)$ vs $(z_i - z_{i-1})$, a straight line with slope $4\omega_n^2/g$ should result. The results for three stable lobe patterns are shown in Fig. 3. In each case the slope of the line was measured and the resonant frequency calculated from it. The actual frequency was determined using a rotation counter and a wrist watch. The values are compared in Table I. The agreement is quite reasonable.

Table I. Comparison of the resonance frequencies of a rotating chain as calculated from the internodal spacing and as measured directly.

Number of nodes (including point of suspension)	Resonant frequency calculated from slope (rad/sec)	Measured resonant frequency (rad/sec)
3	25.8	23.2
4	33.3	34.0
5	48.8	47.1

APPARATUS

A centripetal force apparatus with a variable speed rotor and revolution counter is particularly convenient for demonstrating this effect.[6] A light necklace chain can be attached to the end of a short rod with super glue. This rod can then be inserted in the coupler which normally holds the mass-spring assembly and secured with a thumb screw. Some models allow the variable speed rotor to be rotated about a horizontal axis so that there is no need to invert the entire apparatus. The entire assembly takes as little as 5 min. With relative ease, one can obtain stable patterns with two to five nodes on a 30-cm chain.[7]

This demonstration is extremely easy to set up. The mathematics is neither completely trivial nor overly complex. Bessel functions are observed by students. Reasonable quantitative results may be obtained quickly. Thus it would seem this demonstration is natural accompaniment to students' first introduction to Bessel's equation.

ACKNOWLEDGMENTS

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[1] J. Satterly, Am. J. Phys. **18**, 405 (1950). The author is indebted to one of the referees for bringing this very nice paper on mechanical oscillations to his attention.
[2] D. A. Levinson, Am. J. Phys. **45**, 680 (1977).
[3] J. P. McCreesh, T. L. Goodfellow, and A. H. Seville, Am. J. Phys. **43**, 646 (1975).

[4]The solution to the time-dependent problem is given in M. M. Smirnov, *Problems on the Equations of Mathematical Physics*, translated by W. I. Wills (Noorhoff, Gronigen, The Netherlands, 1966), p. 30.

[5]M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions*, National Bureau of Standards, Applied Mathematics Series 55 (U.S. GPO, Washington, D.C., 1970), p. 30.

[6]For example Cenco 7435-001 Motor-Driven Variable Speed Rotor.

[7]Some care must be taken to approach resonance from the low frequency side. The chain can be adiabatically drawn into very large amplitude patterns by passing slowly through resonance to higher frequencies. These large amplitude patterns are not stable against perturbations. An easy check to see if true resonance frequency has been exceeded is to briefly straighten the chain; true resonances are self-starting.

[8]An interesting example of color addition was produced by taking three flash pictures of the spinning chain on the same frame of color film. For each flash a different color filter was used. In each exposure a single color line image was produced except at the nodes where the color addition resulted in a white image.