CSUC
Department of Physics
Class Notes

Pre-Introduction to ***Vector Geometry***

I. *INTRODUCTION*

We wish to study ordinary *Geometry* by using "Vector" techniques. Your first response may well be " why ? " ... don't I already ***know*** Cartesian geometry? The answer is, of course, "yes", and in fact many of you already know geometry not just one but ***two*** different ways: its Cartesian formulation and its Euclidean formulation. The vector formulation that arrived about 1900 already represented a *third-wave* "Gibbsian" expression of something that had been well studied for *thousands* of years. So, you may justifiably ask "what gives"? You will be even more dismayed to learn that, even as we speak, a *fourth-wave* expression of geometry is being formulated and is well underway to supplanting the other three! So are we (you may ask ...) proposing to learn something we already *know* and which, in fact, is going out of date even before we start? The answer is a resounding **yes!** The reason is deep and abiding and goes as follows. As we proceed through the mathematical sciences, one of the greatest lessons is the realization that how we choose to *organize* our knowledge is of the *greatest possible importance* in our ability to reason about it. We come to realize that the very *same thing* can be said and understood **MANY WAYS**. Some organizations of knowledge reveal and expose assumptions and relationships that would **never** have occurred to us otherwise. Likewise, some symbolic representations of knowledge allow vastly more efficient manipulation and reasoning. An archetypal example of this is the comparison between the Roman Numeral representation of numbers and the Decimal (Hindu/Arabic) representation (e.g. just try multiplying numbers using Roman Numerals ... it will drive you batty!). We will experience the same sort of thing with the Vector formulation of geometry. Beyond this, Geometry in its Vector formulation also serves most students as the pathway to *yet more abstract* mathematics. So you need to learn it not just because it is extremely powerful (which it is ...and, frankly, many folks decide to go no further) but because, later, should you go on, you will "deconstruct" it yet more and your mind will see the pathway forward into **really really** powerful and ever more abstract formulations. The "ah hah!" moments will come crashing in non-stop. This "Pre-Introduction" is intended as both motivational and as the tiniest preview of the kinds of things we will encounter in our study of Vectors. Don't worry that the details aren't here yet ...that will happen soon enough!

It may seem strange to you that we have the freedom to simply "*formulate*" a new way of doing things. In the past you probably imagined that mathematics was *done* or *complete* and it was your task simply to "learn it". However, modern mathematics often (actually, it's pretty relentless ...) undertakes to look at old things *new* ways. The "game" is to construct *logical systems* by merely proposing a set of *elements* that obey a smallish, cleverly chosen, specific set of rules. The elements needn't have any meaning whatsoever or any other reference (e.g. to the "physical world"). As we study "Geometric Vector Spaces" we are actually using one such system. Although we do, indeed, intend to model the three-dimensional *"Physical-Space"* we actually live in, the underlying structure can also, amazingly, be applied to a wide variety of other physical systems. The point of doing this is that we are made, thereby, very much aware of precisely which properties (*axioms-propositions-postulates etc.*) are responsible for which specific outcomes. We have "penetrated" the inner meaning of things, so to speak, and we see what is really going on. Then, as we add more structure, we narrow the applicability of our resultant system. It becomes ever more specific. In our development here we shall start with the very most elementary level of structure and then add on step by step. For those of you who appreciate the "deep messages for geometry" going on here, the *moment of truth* happened near the end of the 18^{th} century when, for the first time in human history, scholars realized that geometries other than *Euclidean Geometry* were in fact possible. This was a **terrible** shock! Because, if there are multiple mutually-competing geometries "out there", then what we thought was the absolute *TRUTH* all along, really has the status of a *model* of the truth and may be an incomplete or partial truth. The ancient Greeks truly believed that mathematical truth and physical truth were inextricably the same thing! We now know better, but it hasn't been easy or pleasant to digest this crowning fact. We moderns, for example, have now internalized that Einstein showed us that the world seems actually to be *Riemannian* (a yet more complicated geometry) and becomes *Euclidean* only when gravity is weak, as it apparently is in our corner of the Universe. Yikes! What we are really doing, then, is building a good mathematical system that may (or may not ...) be a good model of the physical world we live in. Mathematics is *always* **logical** truth, but it need not be **physical** truth (ouch!). This is an essential realization the ancient Greeks never had.

Let us begin the journey.

II. *EARLIER FORMULATIONS*

1.) *Euclidean Geometry*

Two and a half millennia ago, the Greeks formulated the first axiomatic presentation of geometry that we know of. You know it as *Euclidean Geometry.* It was a monumental achievement and served the world as the absolute *prototype* of logical thinking until almost the present day. Everyone who was anyone had to master it. It had a very rigorously limited set of *things* to build from (nothing else is allowed) and a well defined set of *rules of reasoning* you had to stick to. The *things* were given a set of properties that were stated in the famous ***5 postulates***. These postulates were just that, i.e. they were statements to be accepted as true without challenge. Then you "go at it" and see what else must follow logically from this starting place. Well, you can prove an awful lot and Euclid filled 13 books full with his proofs including the Pythagorean theorem and tons more. Over the following centuries any claim to academic fame usually included proving yet more stuff. What is important to us is not the content of Euclidean Geometry (all of which you pretty much know), but rather the *manner* in which it is done. First off, the *things* you get to talk about are: points, line segments, lines, circles and angles. Note especially, that this is ***ALL*** you get to draw! There are no coordinate axes and no units, no origin as a reference spot, and most surprisingly ... almost no **numbers**!!! Nothing (but nothing!) is allowed on the page except the very things you are talking about. Also, there is no *Algebra* since that hadn't been invented yet. It comes as a real shock to most modern people that you could prove anything at all! And yet, the Greeks learned far (far!) more about the geometry of the physical world than most people on today's sidewalks will ever know! So who's laughing now? I would like to call their mode of reasoning "*Absolute*". Only *Absolute* objects are allowed in the discussion and only their *Absolute* properties can be used in reasoning. The resultant discussion is exquisitely **visual** and the reasoning is astoundingly transparent and concrete to all participants. So...what's not to like? The answer for two thousand years was "nothing"! It was only in the seventeenth century, with the development of algebra, that the power of symbolic *numerical* reasoning was appreciated by a wider audience. The way was prepared for the second formulation. The intrigue here was the "temptation" to turn Geometry "into" Algebra, and then to use the power of Algebra to easily prove things that are really tough to show in the plain old *Euclidean* way.

I close here with an insider's comment about an academic debate for the ages. For those of you who like monumental dramas/slugfests that extend across the centuries and millennia, you can't do better than the struggle between viewing the world as being essentially **Geometric** (and thus spatial/visual, so to speak, i.e. *Euclidean/experiential/constructive/Pascal-like*), on the one hand, or rather as **Algebraic** (and thus grammatical/verbal, so to speak, i.e. *Cartesian/rational/analytic/Descartes-like*) on the other. Wowee...what a fight!

Who says Math and Physics aren't cultural disciplines?...! We turn now to this second approach.

2.) *Cartesian Geometry*

Cartesian geometry was probably "the" way you learned geometry at all. You are most likely unaware that most of humanity for most of history did it quite differently! In fact, Cartesian Geometry is the complete "flip-side" approach to that of Euclidean Geometry. First off, it deals almost exclusively in ...yup... **numbers**! What is stranger yet (to the Greeks that is), in order to get those numbers into the discussion we have to introduce a whole bunch of things that *aren't even there.* What are these? Well, there are those coordinate systems mainly. When you look out the window on the world, you do **NOT** see coordinates axes painted on the ground (they are, in fact, complete fictions)! Nor is there an origin and nobody has chosen units. You may be giving me a bewildered look just now and wish to protest..." but what's the harm? ". As we will soon discover, there are many serious failings and pitfalls here - the harm is *grievous.* What was gained, of course, was the staggering power of algebra. So powerful was this that the great shortcomings were pretty much overlooked for a couple hundred years. But that's over now - and has been for a century. The first (and lesser ... though still big) objection is that in turning geometry into algebra, we have completely lost any "visual" connection to our discussion. The greater objection is that, in introducing things into any discussion of *physical law* that aren't really there, we end up with statements that are a hopelessly mixed up tossed-salad of fact and fiction and we can't easily discern which is which! We lose sight of just which structure (the real or the fake one) gives rise to any specific outcome. We lose track of what is arbitrary and what is essential. The words we will use are *Absolute* and *Relative. Absolute* things don't depend in any way on the arbitrary elements in our discussion, whereas *Relative* things "relate back" in some way to arbitrary things. Since geometric relationships are not arbitrary in any way, we need to remove these relative things if we are to clearly recognize what is essential and necessary. You may wish to visualize Cartesian axes and all other relative things as a sort of "scaffolding" that we introduced to complete the argument, but then must be removed once the logical structure is complete. The introduction of the Vector formulation of geometry was an enormous step in this modern direction, and we need to do it (just as we also need to take pity on all those classical Greeks who are spinning in their graves...).

The Cartesian formulation of Geometry starts with the concept of *point location*. You are all familiar with the notion of a "coordinate triple" of numbers e.g. $(5, 2, -3)$ which specifies a point-location in space. Of course, the numbers involved would be meaningless without their reference back to established coordinate axes and an origin (plus units of measurement). The problem here is that the choices of origin, axis orientation and units are all *completely* arbitrary. But geometric relationships are not arbitrary ...ever. Somehow, in some mysterious way, all the *arbitrariness* "cooks out" somewhere in the argument. Most likely, you were never made aware of just how and where this was happening. The Vector formulation of Geometry takes acute note of this, however, and seeks to "build in" these magical steps into the very starting structure of the formulation. These "magic" maneuvers did not come out of thin air, by the way. They were well known to the founders of Cartesian Geometry and employed merely as "insiders' tricks", so to speak. What may surprise you is that there are just three of them and the ***totality*** of geometry may be built out of them ... and only them (we can prove this!). Each of these three maneuvers/structures will receive its own name, which are: **1)** *displacement*, **2)** *dot product*, **3)** *cross product*. We will discuss these in great detail in later sections, but let me just mention here the amazing simplicity of the actual founding ideas and where they come from, so that later discussions don't seem so arbitrary. Let us start with the observation that only differences of point coordinates will ever actually show up in our discussion. Why is this? The deep answer is that, it **has** to be that way because then the arbitrarily chosen *origin of coordinates* drops out of the problem! You may never have noticed this. This idea of a **difference** is so important that we have a special symbolism for it, the "delta" notation e.g.: $\Delta x \equiv (x_2 - x_1)$. You are supposed to notice that if you move the origin of coordinates up or down the x axis, all x coordinates are shifted by an equal amount that cancels on taking a difference. *Differences* of location are then seen to be, in a deep way, more fundamental than location itself, and all physical laws will be expressed in terms of them. What a surprise this is! We give the name *displacement* to a **change** of location and it will be the founding concept in our building of the vector formulation of geometry. Indeed, the very word *Vector* comes from the Latin word for "vehicle of transport". A vector carries you from one location to another, so to speak. We will need three things to be a "vector displacement": **a)** amount of travel ($\mapsto$ *magnitude*), **b)** axis of travel, and **c)** sense of travel up or down the axis ($\mapsto$ *direction*). People often summarize these by saying that a Vector has magnitude and direction. Next, we will need to get rid of the (arbitrary) choice we made in setting up our axis *directions*. The (only!) way to do this is to use what are known as the cosine and sine "double-angle formulas", which we list next (and will prove simply later).

$$cos(\theta - \phi) = cos(\theta)\,cos(\phi) + sin(\theta)\,sin(\phi) \tag{1}$$
$$sin(\theta - \phi) = sin(\theta)\,cos(\phi) - cos(\theta)\,sin(\phi) \tag{2}$$

These relationships will do for angles what we just did for coordinates, namely they give us **differences**! By this simple stratagem, then, we will eliminate our choice of axis directions. Only **differences** of location and angles will have any significance and they will recreate for us the Greek idea of "Absolute Things" which won't depend on arbitrary choices. So, when we pick up our discussion of *Vector Geometry*, as we do next, be prepared to notice the singular importance of displacements, dot-products, and cross-products. The real miracle is that **ALL OF GEOMETRY** is contained in them.

III. *BASIC VECTOR MANIPULATION: ...ADDITION AND SUBTRACTION*

(This material is forthcoming and is currently represented in alternative notes)

IV. *THE DOT PRODUCT AND CROSS PRODUCT*

1.) *The Origin of the Dot and Cross Products*

The "dot product" has its origin in a simple Cartesian construction. Consider the following 2-dimensional picture:

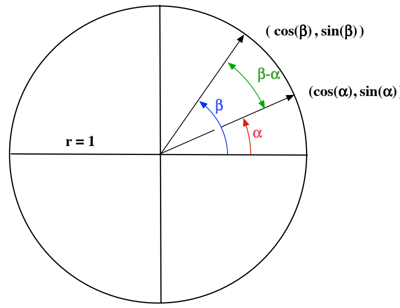

FIG. 1: The Setting for the Dot-product.

Here we see two vectors in a plane drawn from the origin out to the unit circle. Since the vectors start at the origin, the coordinates of the tips of the vectors are also, then, the components of each of the vectors. So we may write:

$$\langle x_1, y_1 \rangle \;=\; \langle cos(\alpha)\,,\, sin(\alpha)\,\rangle \equiv \hat{r}_1$$
$$\langle x_2, y_2 \rangle \;=\; \langle cos(\beta)\,,\, sin(\beta)\,\rangle \equiv \hat{r}_2$$

You are supposed to notice that the two angles $\{\alpha, \beta\}$ refer back to the $\hat{x}$ axis, and thus are *relative* to something "artificial" (i.e. the orientation of the $\hat{x}$ axis). However, if we compute the following simple combined product:

$$x_1\,x_2 \,+\, y_1\,y_2 \;=\; cos(\alpha)\,cos(\beta) \,+\, sin(\alpha)\,sin(\beta) \;=\; cos(\beta - \alpha)$$

by the cosine double angle formula, the result depends only on the **difference** of the angles and *that* is **independent** of the orientation of the $\hat{x}$ axis. So! By this maneuver we have removed any dependence on how the axes are tilted. You can immediately see that we can create a second and similar outcome by computing:

$$x_1\,y_2 \,-\, y_1\,x_2 \;=\; cos(\alpha)\,sin(\beta) \,-\, sin(\alpha)\,cos(\beta) \;=\; sin(\beta - \alpha)$$

where, now we have invoked the sine double angle formula (we will use this second result in just a moment). We have, now, merely to recognize that *any* vector can be written as its magnitude times its unit vector direction e.g.

$$\vec{r} = \parallel \vec{r} \parallel\; \hat{r} \;=\; \parallel \vec{r} \parallel\; \langle cos(\alpha)\,,\, sin(\alpha)\,\rangle \;=\; \langle x, y \rangle$$

and now we are prepared to write our dot product in its final form.

$$x_1\,x_2 \,+\, y_1\,y_2 = \parallel \vec{r}_1 \parallel \parallel \vec{r}_2 \parallel \,(\, cos(\alpha)\,cos(\beta) \,+\, sin(\alpha)\,sin(\beta)\,) = \parallel \vec{r}_1 \parallel \parallel \vec{r}_2 \parallel\, cos(\beta - \alpha)$$

Actually, we usually do it "one better" by calling the angle $\beta - \alpha \,\equiv\, \theta_{included}$ on the right hand side. In summary, then, we will now exploit our understanding from above and simply *define* the dot product $\vec{r}_1 \cdot \vec{r}_2$ of two vectors as:

$$\vec{r}_1 \cdot \vec{r}_2 \;\equiv\; x_1\,x_2 \,+\, y_1\,y_2 = \parallel \vec{r}_1 \parallel \parallel \vec{r}_2 \parallel\, cos(\theta_{included}) \tag{3}$$

Notice especially that the Cartesian components in the middle expression, i.e. $\{\,x_1,\, x_2,\, y_1,\, y_2\}$ all **do** *individually* depend on the orientation of the axes, but that the complete combination used gives an output which **does not** ! You can use any axes you please and the outcome will always be the same. That's why it's common and mature practice to simply write $\vec{r}_1 \cdot \vec{r}_2$ and never mention the axes at all. Of course we **do** use axes all the time, but we are not

bound to using any *particular* ones as long as we use the dot product "recipe". The left hand side of this expression uses only "whole-vectors" and might be called a "holistic" expression, though I will call it *Absolute*. In reality, it is a "Greek-like" expression such as we have sought all along - nothing artificial! The right hand side is actually what I would call "Polar" in form, and is also exclusively built out of *Absolute* entities...so it is also "Greek-like".
Now, we mentioned (above) that we would later use the companion relation using the sine double angle formula and I include it here in its final form so we can talk about it below. It will form the essential innards of what we will eventually call the "cross-product".

$$x_1\, y_2 \,-\, y_1\, x_2 = \|\, \vec{r}_1 \,\|\|\, \vec{r}_2 \,\|\; sin(\theta_{included}) \tag{4}$$

2.) *What Do the Dot and Cross Products "Give Us" ?*

We have just spent a lot of time developing things that are independent of axes etc. , i.e. that are *Absolute*. But these things had better be **useful** to us or what we have just done will be *Absolutely Stupid* ! The deep insight is that "Absolute" and "Useful" go hand in hand. For simplicity's sake, let us keep our discussion, just for the moment, in a 2-D plane. By definition, any vector has *Magnitude* and *Direction*. So for any two vectors $\vec{A}$ and $\vec{B}$ it doesn't take long before the natural question arises: "So ... just *how much* are they in the *same* direction and how much are they not ?" The fancy words meaning *"same direction"* and *"not same direction"* are, of course, **parallel** and **perpendicular**. Now in fact, if we look back at equations 3) and 4), we readily observe that they give us, as output, ***precisely*** the *product* of **parallel parts** and the product of **perpendicular parts** ! This is spectacular... and will, in short order, provide us with all the information we need to "do" geometry.

Let us now collect our results more systematically. Given any two vectors $\vec{A}$ and $\vec{B}$, ask: "how much of $\vec{B}$ falls along the direction of $\vec{A}$?" The answer is $\|\, \vec{B} \,\|\; cos(\theta_{included})$, and the vocabulary word we'll use is *Perpendicular Projection* or, more commonly, just *Projection*. The following picture displays pretty well just what we mean.

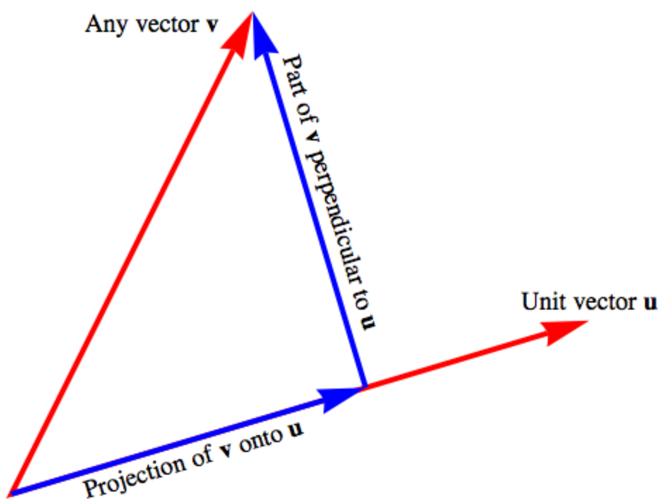

FIG. 2: The Idea of Projection.

If we now use the symbolism $B_{\|}$ for the component of $\vec{B}$ along the direction of $\vec{A}$, then:

$$B_{\|} \,=\, \vec{B} \cdot \vec{A} \,/\, \|\, \vec{A} \,\| \tag{5}$$

The unit vector in the direction of $\vec{A}$ is $\hat{A} \,\equiv\, \vec{A} \,/\, \|\, \vec{A} \,\|$, just as the unit vector in the direction of $\vec{B}$ is $\hat{B} \,\equiv\, \vec{B} \,/\, \|\, \vec{B} \,\|$.
Using these nice notations we can write, succinctly:

$$\begin{aligned} B_{\|} \;&=\; (\vec{B} \cdot \hat{A}) = \|\, \vec{B} \,\|\; cos(\theta_{included}) \\ A_{\|} \;&=\; (\vec{A} \cdot \hat{B}) = \|\, \vec{A} \,\|\; cos(\theta_{included}) \end{aligned}$$

The next two figures bring these ideas all together. We can now display a set of equivalent results:

$$\vec{A} \cdot \vec{B} = \|\, \vec{A} \,\|\; B_{\|} \;= \|\, \vec{B} \,\|\; A_{\|} \;= \|\, \vec{A} \,\|\|\, \vec{B} \,\|\; cos(\theta_{included}) \;=\; A_x B_x + A_y B_y \tag{6}$$

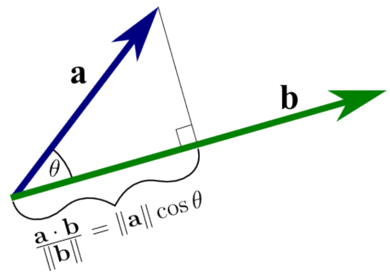

FIG. 3: Projection as Given by the Dot-Product.

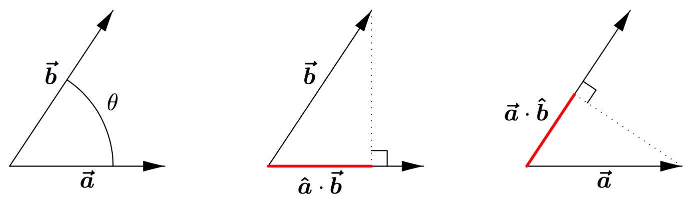

FIG. 4: Vectors projecting onto each other.

Further, since any vector lies entirely along itself, we can write:

$$\vec{A}\cdot\vec{A} = \parallel \vec{A} \parallel^2 = A_x A_x + A_y A_y \; = \; A_x^2 + A_y^2 \; = \; A_\parallel^2 + A_\perp^2 \tag{7}$$

where parallel and perpendicular components can be referenced to *any* direction $\hat{B}$ whatsoever.

The Cross product seems to give students more trouble than the Dot product and, to be sure, it leads to a rather richer and more involved discussion. The first thing to remember though is that, if we stay in a *plane*, both of these constructions are simply the sine and cosine double-angle formulas in new clothing. As we have just said, the dot product carries the geometric meaning of "**PROJECTION**" (*or product of parallel parts*). We now assert that the cross product will, in its turn, carry the important geometric concept of "**AREA**" (*or product of perpendicular parts*). As we will soon come to see ... these two combinations exhaust the possibilities of interesting things to say and this is why all of Geometry is contained in them. Between these two constructions, we can learn everything. Consider the following picture which depicts two non-parallel vectors $\vec{w}$ and $\vec{v}$. Notice, especially, that these vectors define a natural parallelogram, and we can find its area with the 2-D version of the "cross-product", as follows.

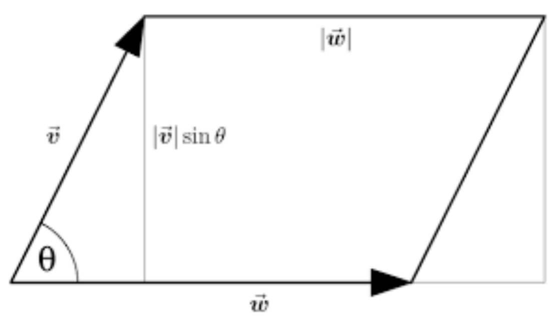

FIG. 5: Vectors defining a Parallelogram.

We have already developed the following relation and listed it in equation 4) above.

$$w_x\, v_y \; - \; w_y\, v_x \; = \parallel \vec{w} \parallel \parallel \vec{v} \parallel sin(\theta_{included}) \; = \; \text{base} \times \text{height} \tag{8}$$

Here you can easily see that our combination has, indeed, produced precisely the **area** of the parallelogram.

3.) *3-Dimensions ... Our Next Challenge*

Physical Space is 3-Dimensional. We now need to generalize our previous results to the forms they will have in that fully 3-D setting. For a single vector $\vec{A}$ how we do this is so straight-forward and well known to you as to scarcely merit attention.

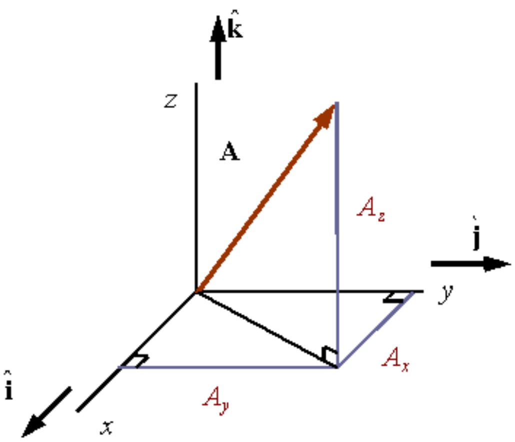

FIG. 6: Vector in 3-D Space

In the figure above we see a typical vector $\vec{A} \equiv \langle A_x,\ A_y,\ A_z \rangle$, where now <u>three</u> components are necessary to specify the vector. This shocks and surprises no one. It also comes as a surprise to no one that the magnitude of the vector is given by:

$$\vec{A}\cdot\vec{A} = \parallel \vec{A} \parallel^2 \,=\, A_x^2 + A_y^2 + A_z^2 \,= \parallel \vec{A}_{\hat{x}-\hat{y}} \parallel^2 + A_z^2. \tag{9}$$

We have made use of the notation $\vec{A}_{\hat{x}-\hat{y}} \equiv \langle A_x,\ A_y,\ 0 \rangle$ which is the "projection" of our $3-D$ vector $\vec{A}$ onto the $\hat{x}-\hat{y}$ plane. What *is* a little surprising (though not shocking ...) is that we have only had to use our $2-D$ results (twice) to achieve a $3-D$ result ! The first use was in using our knowledge of $\parallel \vec{A}_{\hat{x}-\hat{y}} \parallel$ from $2-D$ geometry, and then drawing a <u>second</u> plane through $\vec{A}_{\hat{x}-\hat{y}}$ and our new direction $\hat{\mathbf{z}}$, which allowed us to use our $2-D$ result again. But now, please take in the logical consequence of all this. The result given in equation 9) is $\Rightarrow$ **independent** $\Leftarrow$ of the choice of axes and their orientations ! If you were to choose <u>any</u> three new (mutually perpendicular) axes, the values of each of the numbers $\{A_x,\ A_y,\ A_z\}$ <u>would</u> change, but $\overline{\text{the}}$ value of the sum of their squares <u>would NOT</u> ! $\vec{A}\cdot\vec{A} \,= \parallel \vec{A} \parallel^2$ is still an *Absolute* quantity. Now comes the shock ... . Suppose we had a *second* generic vector $\vec{B} \equiv \langle B_x,\ B_y,\ B_z \rangle$. We now get to assert that:

$$\vec{A}\cdot\vec{B} \,\equiv\, A_x\,B_x \,+\, A_y\,B_y + \,A_z\,B_z \,= \parallel \vec{A} \parallel \parallel \vec{B} \parallel cos(\theta_{included}) \tag{10}$$

gives an *Absolute* quantity just as before, and indeed with precisely the same meaning! So **WHY** do we get to assert this ? Well ... just consider. Since **any** vector dotted with itself is an *Absolute* quantity, that means $\vec{A}\cdot\vec{A}$ and $\vec{B}\cdot\vec{B}$ are both *Absolute*. Yes, but then so is $(\vec{A}+\vec{B})\cdot(\vec{A}+\vec{B})$. And now we can write:

$$(\vec{A}+\vec{B})\cdot(\vec{A}+\vec{B}) \,=\, \vec{A}\cdot\vec{A} \,+\, \vec{B}\cdot\vec{B} \,+\, 2\,\vec{A}\cdot\vec{B} \tag{11}$$

In fact, by moving all but the last term to the left hand side, we have:

$$(\vec{A}+\vec{B})\cdot(\vec{A}+\vec{B}) \,-\, \vec{A}\cdot\vec{A} \,-\, \vec{B}\cdot\vec{B} \,=\, 2\,\vec{A}\cdot\vec{B} \tag{12}$$

and since the left hand side is manifestly *Absolute* the right hand side must be too! And, further, since we could always draw our axes so that $\vec{A}$ and $\vec{B}$ fall in the $\hat{x}-\hat{y}$ plane, the meaning can't have changed either. This is power.

At this point we could start deriving zillions of useful results, but since these notes are intended to be principally motivational, I won't just yet ... except for two.

First, since we now have, as a central conclusion independent of *any coordinate axis frame*, that:

$$\vec{A} \cdot \vec{B} = \| \vec{A} \| \| \vec{B} \| \, cos(\theta_{included})$$

we can conclude that two vectors are *perpendicular* ... **if and only if** their dot product is zero... because then $cos(\theta_{included}) = 0$, and that implies $\theta_{included}$ must be 90°.

Second, we recognize that the components of any vector $\vec{A}$ are really just projections onto the coordinate axes and are thus, themselves, just dot products, e.g. :

$$\begin{aligned} A_x &= \vec{A} \cdot \hat{x} \\ A_y &= \vec{A} \cdot \hat{y} \\ A_z &= \vec{A} \cdot \hat{z} \end{aligned}$$

So now we may write a very attractive summary conclusion:

$$\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z} = (\vec{A} \cdot \hat{x})\hat{x} + (\vec{A} \cdot \hat{y})\hat{y} + (\vec{A} \cdot \hat{z})\hat{z} \tag{13}$$

Our final challenge is to extend the notion of *Area* to three dimensional space. This isn't hard to visualize. We start with the (previously mentioned) observation that any two non-parallel vectors, say $\vec{A}$ and $\vec{B}$, determine a natural *Parallelogram* in a natural plane. Only now, while still remaining flat, the plane can be oriented or tilted any which way. The first realization we need is that the orientation of any plane is uniquely characterized by a single vector, its **normal** or perpendicular, designated $\hat{n}$ in the next picture. Since, by assumption, the vectors $\vec{A}$ and $\vec{B}$ are in the plane, they must both be perpendicular to the normal.

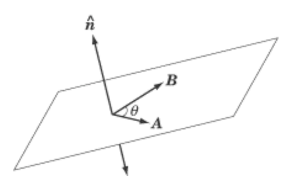

FIG. 7: Two Vectors in 3-D Space

The second realization we need is ... that <u>areas</u> can *project* onto other areas just as <u>vectors</u> can *project* onto other vectors.

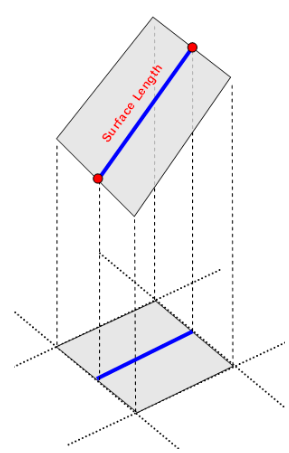

FIG. 8: Areas Project Onto Each Other

And now for the <u>best</u> realization of all. Just as with vectors... the amount of projected area is the original area times the cosine of the angle between the areas ... and that angle is also the angle between the normals!

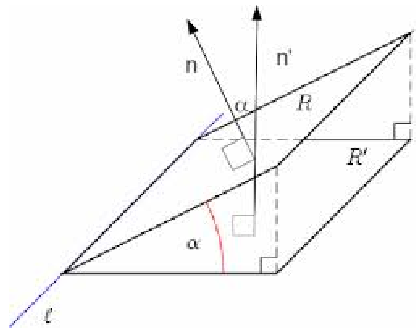

FIG. 9: Areas and their Normals

For this diagram, then, we write: $\mathbf{R}' = \mathbf{R}\, cos(\alpha) = \mathbf{R}\, \hat{n} \cdot \hat{n}'$.

And if we now put all these pieces together,we get a picture like the following one where we have drawn two vectors and the Parallelogram they define and its normal all positioned in 3-D space.

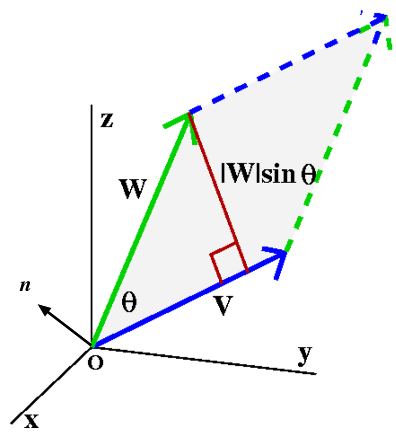

FIG. 10: Oriented Parallelogram

The parallelogram has an area I'm going to call A which is given by: $A = \|\vec{V}\| \, \|\vec{W}\| \, sin(\theta)$. It also has a normal vector $\hat{n}$. Both of these geometric entities are *Absolute* ... and we use them to *define* the Vector Cross Product ! That is, we define a new vector that is designated by the symbols $\vec{V} \times \vec{W}$ by the following *Absolute* prescription:

$$\vec{V} \times \vec{W} \equiv A\, \hat{n} \tag{14}$$

By the way, the ordering of the vectors is crucial and we specify that $\{\vec{V}, \vec{W}, \hat{n}\}$ form a "right-handed triple" just as $\{\hat{x}, \hat{y}, \hat{z}\}$ do. So be careful: $\vec{W} \times \vec{V} = -A\, \hat{n}$! At this point, many of you may wish to protest (I know *I* did!) . At the outset of any problem, we will often know our information about $\vec{V}$ and $\vec{W}$ only in *Cartesian* form! But then, how do we find $\hat{n}$ and A ? Well, get ready for the *Magic* of mathematics. Just suppose then, that it is so, and we have specified our starting vectors, as is so often the case, by known components:

$$\begin{aligned} \vec{V} &\equiv \langle V_x, V_y, V_z \rangle \\ \vec{W} &\equiv \langle W_x, W_y, W_z \rangle \end{aligned}$$

We know that these vectors project onto the $\hat{x} - \hat{y}$ plane, giving:

$$\begin{aligned} \vec{V}_{xy} &\equiv \langle V_x, V_y, 0 \rangle \\ \vec{W}_{xy} &\equiv \langle W_x, W_y, 0 \rangle \end{aligned}$$

And we also know that $\vec{V}_{xy}$ and $\vec{W}_{xy}$ in the $\hat{x} - \hat{y}$ plane define their own "projected" parallelogram there with an area given by: $V_x W_y - V_y W_x$. And finally, since we also know that the $\hat{x} - \hat{y}$ plane has **its** normal vector in the $\hat{z}$ direction, we can "project" the area $\vec{V} \times \vec{W} = A\hat{n}$ ***onto*** the $\hat{x} - \hat{y}$ plane and conclude it must be the case that:

$$(\vec{V} \times \vec{W}) \cdot \hat{z} = A\, \hat{n} \cdot \hat{z} = V_x W_y - V_y W_x$$

Actually, we can simply permute the axis symbols to collect all three components.

$$\begin{aligned} (\vec{V} \times \vec{W}) \cdot \hat{x} &= A\, \hat{n} \cdot \hat{x} = V_y W_z - V_z W_y \\ (\vec{V} \times \vec{W}) \cdot \hat{y} &= A\, \hat{n} \cdot \hat{y} = V_z W_x - V_x W_z \\ (\vec{V} \times \vec{W}) \cdot \hat{z} &= A\, \hat{n} \cdot \hat{z} = V_x W_y - V_y W_x \end{aligned} \tag{15}$$

But wait ... (!) ... once we know the dot product of a vector with all the coordinate axes ... we know its components! We conclude:

$$\vec{V} \times \vec{W} \equiv A\hat{n} = \big\langle (V_y W_z - V_z W_y), (V_z W_x - V_x W_z), (V_x W_y - V_y W_x) \big\rangle \tag{16}$$

Further, since $\hat{n}$ is a unit vector and $\|\hat{n}\| = 1$, we must have: $(\hat{n} \cdot \hat{x})^2 + (\hat{n} \cdot \hat{y})^2 + (\hat{n} \cdot \hat{z})^2 = 1$, we can then also conclude that:

$$\|\vec{V} \times \vec{W}\| = A = \sqrt{(V_y W_z - V_z W_y)^2 + (V_z W_x - V_x W_z)^2 + (V_x W_y - V_y W_x)^2} \tag{17}$$

You can see that the "Cross Product" $\vec{A} \times \vec{B}$ of two vectors can be understood (and used!) in several distinct but perfectly equivalent ways. We can use Cartesian components or the more abstract *Absolute* notation which never even mentions components. The "holistic" or *Absolute* notational way is usually less familiar and less comfortable to younger students, whereas the Cartesian way seems more "concrete". Abstract manipulation often feels "wierd". With a bit of practice, however, this mental dichotomy goes away and you will use *whatever* representation seems most natural in the moment. We use them both. Let me summarize next, then, the Cross-Product properties in their *Absolute* forms:

1. $\vec{A} \times \vec{B}$ is itself a vector and is **linear** in each vector of the product (we say it is *bilinear*) so that e.g. $\vec{A} \times (\vec{B} + \vec{C}) = (\vec{A} \times \vec{B}) + (\vec{A} \times \vec{C})$ *etc.* .
2. $\vec{A} \times \vec{B}$ is perpendicular to both $\vec{A}$ and $\vec{B}$, so that $\vec{A} \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\vec{A} \times \vec{B}) \equiv 0$
3. $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$ so that $\vec{A}$, $\vec{B}$, and $\vec{A} \times \vec{B}$ form a right-handed triple of vectors.
4. $\| \vec{A} \times \vec{B} \| = \| \vec{A} \| \| \vec{B} \| \; sin(\theta_{included})$
5. $\vec{A}$ and $\vec{B}$ are *parallel* ... **if and only if** $\vec{A} \times \vec{B} = 0$ so ... $\vec{A} \times \vec{A} \equiv 0$.

It will be highly gratifying to observe, at this point, that **all** of our Cartesian component results - for <u>both</u> the dot and cross products - pop out of these *Absolute* relations above, if only we apply then to the unit axis direction vectors $\{\hat{x}, \hat{y}, \hat{z}\}$. We need only remember that the dot product gives us "*projection* = product of parallels" and the cross product gives us (parallelogram area)·(unit normal direction vector $\hat{n}$). We list these unit vector relations next.

$$\begin{aligned} \hat{x} \cdot \hat{x} &= 1 \\ \hat{y} \cdot \hat{y} &= 1 \\ \hat{z} \cdot \hat{z} &= 1 \\ \hat{x} \cdot \hat{y} &= 0 \\ \hat{y} \cdot \hat{z} &= 0 \\ \hat{z} \cdot \hat{x} &= 0 \\ \hat{x} \times \hat{y} &= 1\,\hat{z} = -\,\hat{y} \times \hat{x} \\ \hat{y} \times \hat{z} &= 1\,\hat{x} = -\,\hat{z} \times \hat{y} \\ \hat{z} \times \hat{x} &= 1\,\hat{y} = -\,\hat{x} \times \hat{z} \\ \hat{x} \times \hat{x} &= 0 \\ \hat{y} \times \hat{y} &= 0 \\ \hat{z} \times \hat{z} &= 0 \end{aligned} \tag{18}$$

We need only take any two vectors specified as $\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$ and $\vec{B} = B_x\hat{x} + B_y\hat{y} + B_z\hat{z}$ and proceed to simply dot them or cross them term by term. Like a miracle, we re-create the component formulae so tediously derived above. The two formulations really are equivalent.

We owe the development and popularization of these techniques to the first great mathematical physicist to ever appear in the United States, Josiah Willard Gibbs of New Haven, Connecticut. The year was about 1881 and he was an eminent faculty member at Yale University who also made enormous contributions in Physics, Chemistry, Thermodynamics and Statistical Mechanics. He also published his own personal notes for **his** students (just such as these!) .. only in those days publishing was a whole lot harder.

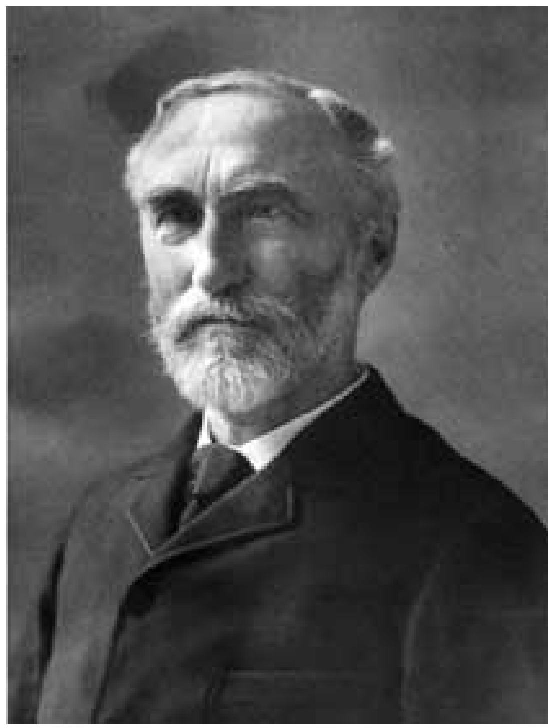

FIG. 11: Josiah Willard Gibbs