CSUC
Department of Physics
301A Mechanics
***PORTFOLIO* Problem 1.)** due: Friday, September 28 Enrico Fermi's *birthday* is 9/29/1901 (Rome)

A. Realistic Projectiles: Your first *PORTFOLIO* problem.

P1.) A projectile is fired straight upwards from the earth's surface experiencing both gravity and an air drag force whose magnitude is of the form: $F_{drag} = \beta \left(e^{\alpha |v|} - 1 \right)$ where α and β are parameters to be determined by the measured initial conditions and asymptotic terminal velocity. This form is actually quite realistic and was deduced as a curve fit to *Remington Arms* ammunition data. The projectile is fired with an initial velocity of $v_0 = 3 \times 10^3 \, ft/sec.$ and initially decelerates at $100\, g$ upon emerging from the muzzle. We estimate the terminal velocity as $v_{terminal} = 50\, m/sec.$

a) Find $v(t)$ analytically for the trips up and down.

b) Find $v_{terminal}$ on the way down from simple examination of the equation of motion and show that it agrees with the conclusions of part a).

c) Compute the trajectory **numerically** and display your results in both tabular and graphic forms. Find how long the projectile takes to reach the top of its trajectory and also the total duration of its flight (in natural units and in seconds!). Argue that your answer is probably correct to one percent accuracy - and how you checked that.

d) Finally, compare the analytic result for $v(t)$ from part **a)** with your numeric results for $v(t)$ from part **c)** by graphing them side by side. Comment !

1. *Philosophical Notes on Problem P1*

This problem has *many* objectives. First, like most (real) problems it is intrinsically numerical - and we need to accustom ourselves to doing numerics and graphing without flinching ... or even noticing, really. At the same time, I hope you recognize that we should understand a problem rather well before we ever attempt a numerical solution. Second, this problem $\Rightarrow$ ***requires*** $\Leftarrow$ choosing *intelligent* natural units. You will find that extracting the values of the modeling constants $\{ \alpha, \beta \}$ requires a little numerical work of its own. This is the way life is. Once you have set up your equation of motion in **active form** you are ready for a spread sheet or Python or "some other" numerical device (I used a programable hand calculator the first time I did this) to increment your way along in time.

2. *The BIG Lessons*

1) In order to perform numerical computation you MUST turn your equation from "Passive" to "Active". This requires "actively" choosing "some sort" of units. *Natural* units will deliver an equation that is far simpler to understand physically and ... ***Well Conditioned*** numerically ... i.e. the numbers won't be exponentially large or small. Choosing *intelligent* natural units is **essential** ...not just convenient ! So what are they? This brings in "Physics". The natural sizes offered by this problem are precisely $v_{terminal}$ and g ...so **use** them! By working "backwards" you will find that you have defined natural length and time units.

2) Newton's 2nd Law is an *Algorithm* for incremental forward stepping in time. Dwell on this ...until you believe it! Differential Equations were "invented" with the advent of Newton's Laws and they lead you to the proper understanding! You are to "increment" your way forward using a spreadsheet (...or Python...or what have you!). Whatever you do ... don't treat numeric tools as a magic wand that gives you the answer but relieves you of the burden of understanding things ...no "*Mindless Mathematica Mental-mode*" here!

3) Please notice that I have asked you to connect a few "analytical" deductions with your numeric deductions. In the end, I would hope you would see both processes as equivalent ways of saying the same thing. In your mind's eye, they should simply be alternative expressions and enhance each other, or indeed *require* each other. I deeply respect the great "Central European" tradition that views analysis without numerics as *sterile* and numerics without analysis as *blind*.

B. "Hands On" Notes for Solving P1

1.) First off, ...let's notice dimensions. It must be the case that β has the dimensions of "Force" since it is one of the force entries in Newton's second law - that's clear. But also, α, which only appears inside the argument of a modeling function, must have the net dimension of an "inverse velocity" since the argument of any physical function must be *dimensionless* ! Please contemplate and convince yourself of this fact.

2.) All good Physics is about "*Competition Between Effects*". The natural sizes offered by this problem are precisely g and $v_{terminal}$ which, then, measure for us the two competing effects of this problem: "Gravity and Air-Drag". The wisdom of Physics is to use the sizes of the problem as your "Natural Comparison Amounts". The mass of the projectile might also interest us ... but as we see in so many gravitational problems, it will be observed to "simply drop out" (why does this always happen ?). This being so, we observe that "Nature" has given us a characteristic velocity and a characteristic acceleration. It may not be what we wanted ... but it's what "Nature" gave us ... so we run with it. We introduce new Natural Units $\{L, T\}$ such that:

$$g = 1 \left(\frac{L}{T^2}\right) \qquad \text{and} \qquad v_{terminal} = 1 \left(\frac{L}{T}\right)$$

Now we "back-solve" and conclude that:

$$L = 1 \left(\frac{v_{terminal}^2}{g}\right) \qquad \text{and} \qquad T = 1 \left(\frac{v_{terminal}}{g}\right)$$

At this point we realize that the argument $(\alpha\, v)$ may be written as $(\alpha\, v_{terminal}) \cdot \left(\frac{v}{v_{terminal}}\right)$. This is proper since now $(\alpha\, v_{terminal}) \equiv a$ now becomes our *dimensionless* model constant and $\left(\frac{v}{v_{terminal}}\right) \equiv v'$ becomes our naturally scaled and *dimensionless* active variable. So also $\left(\frac{t}{T}\right) \equiv t'$. In the same way, if we divide our equation of motion by mg, we arrive at a dimensionless version of our second model constant viz. $\beta\,/mg \equiv b$. As our equation for the upward motion, you should be able to arrive at:

$$\frac{d\,v'}{d\,t'} = -1 - b\left(e^{a\,v'} - 1\right)$$

3.) Solving this analytically is really not so hard. First, tidy up (and leave off the primes!):

$$\frac{d\,v}{d\,t} = (b-1) - b\,e^{a\,v}$$

Now, multiply the whole equation by $-a\,e^{-a\,v}$, resulting in:

$$-a\,e^{-a\,v}\frac{d\,v}{d\,t} = -a\,(b-1)\,e^{-a\,v} + a\,b$$

This makes the left hand side a total derivative! We achieve:

$$\frac{d\,e^{-a\,v}}{d\,t} = -a\,(b-1)\left(e^{-a\,v} - \frac{b}{b-1}\right)$$

...and finally ...

$$\frac{d\left(e^{-a\,v} - \frac{b}{b-1}\right)}{d\,t} = -a\,(b-1)\left(e^{-a\,v} - \frac{b}{b-1}\right)$$

And now, if we define $\left(e^{-a\,v} - \frac{b}{b-1}\right) \equiv w$ ...we have:

$$\frac{d\,w}{d\,t} = -a\,(b-1)\;w$$

And this has the simple solution:

$$w(t) = w_0\,e^{-a\,(b-1)\,(t\,-\,t_0)}$$

Now just re-insert the symbol definitions and we are done! The journey downward is just as easy.

4.) Our last task before numerical integration is finding $\{a, b\}$. We have two bits of information to work with. First, at the initial moment $\frac{dv'}{dt'} = -100$ while the velocity is a known value v'_{init}. On the way down, the system rapidly approaches zero acceleration and $v \rightarrow v_{terminal}$ and this means $v' \rightarrow -1$. Using the equation of motion twice with these specifications gives us two algebraic equations for the two unknowns $\{a, b\}$. Of course, these equations are spectacularly non-linear! But who cares anymore ... we have graphing and numerical tools galore! With sensitive graphing etc., we can easily see what values make these two equations true at the same time. That's the solution! This is the attitude you are to adopt. Go at it!

5.) Now possessing $\{a, b\}$ as known definite values, we can apply any technique we wish to increment our way along and find the trajectory at any given time. The simplest technique is Euler's method:

$$\Delta x \equiv \frac{dx}{dt}\Delta t \qquad \text{and} \qquad \Delta v \equiv \frac{dv}{dt}\Delta t$$

where $\frac{dx}{dt} = v$, and $\frac{dv}{dt}$ is given by Newton's second law.

The universal algorithm is then given by:

$$x_{new} = x_{old} + \Delta x \qquad \text{and} \qquad v_{new} = v_{old} + \Delta v$$

The problem then increments itself along. Once you define Δt ...you simply move from increment to increment adding up the changes as you go along. Think of this, as we discussed in class, as filling in a spreadsheet row by row. It is not at all important that Δt be the same for each step ... only that it be "small enough" that the curve be "sensibly flat" over the interval. Thus, for example, once the projectile has reached terminal velocity ... its value is unchanging and completely flat no matter how large the interval is. Sensitive numerical evaluation always exploits such features.