CSUC
Department of Physics
301A Mechanics
***PORTFOLIO* Problem 2.)** due: Monday, December 3, 2018

I. FOURIER SERIES AND THE RESPONSE OF A S.H.O. TO A PERIODIC DRIVE.

A. Introduction

Your second PORTFOLIO exercise is to compute the steady-state response of a damped Simple Harmonic Oscillator to a periodic driving force. This is essentially problem 5.53 on page 214 in your Text, but **here** we will be using an odd "Saw-Tooth" driving force (shown below) instead of the even function shown there. I also want you to use a more realistic (and universally expressed !) set of intervals and damping constants so that you can see how you always scale them out in any real problem. A nice picture of the driving force we are talking about is shown next. Observe how we have centered our coordinate system both horizontally and vertically.

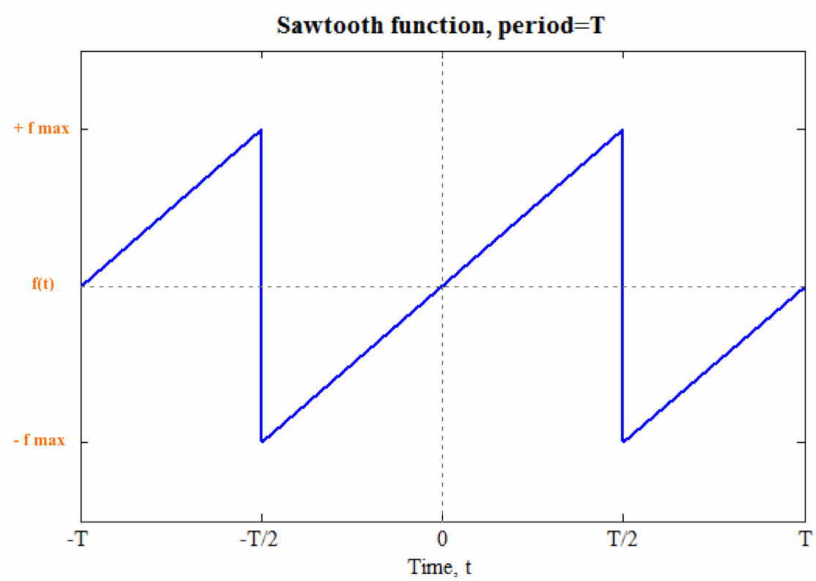

FIG. 1: The driving force for our Simple Harmonic Oscillator

The driving force repeats over a total time interval we'll call T_{drive} (but called simply T on the graph). The Simple Harmonic Oscillator, itself, has a *natural* ***undamped*** period we will call T_o. It will often be more convenient to refer to these sizes through their associated frequencies. We define: $\omega_o \equiv 2\pi/T_o$, and $\omega_d \equiv 2\pi/T_{drive}$. Finally, we will find it very convenient to define their ratio too: $\overline{\omega_d} \equiv \omega_d/\omega_o$. The general outline of this Portfolio Problem will be to: $\mathbf{1}^{st}$) find the Fourier coefficients of this driving force and plot the sum as a Fourier Series up to the first ten non-vanishing terms, then $\mathbf{2}^{nd}$) you will compute the Steady State Response of our S.H.O. to this force. You will actually evaluate the response for four different settings, namely for two different values of $\overline{\omega_d}$ (one less than and one larger than unity) ... and each of these at two different values of the quality factor Q (namely low and high). You will then make tables of the response coefficients and phase shifts and comment on the convergence pattern as was done in example 5.5 in your text. All of this will be laid out in detail in **Part C.** below.

It is not possible to *over-emphasize* the importance of the Fourier approach! It appears everywhere in physics and exposes the most intimate inner workings of Nature. I urge you set up your algebra very very carefully before you do the numerics. Then, the actual evaluation will be rather simple and repetitive. Note especially the rapid convergence of the sum and the selectivity of the response. You may use a simple EXCEL spread sheet or something more sophisticated like Python. The use of MATHEMATICA works well too.

B. Let's Get Started ...

1. The Reading

Please read (or re-read as may be ...) the following sections in your Text.

1.) "Complex Solutions ..." found on pages 181-192.
2.) "Fourier Series" section 5.7 found on pages 192-197.
3.) "Driven Oscillator" section 5.8 found on pages 197-203.
4.) "Fourier Series Notes" These are found on our class web-site and are some great review notes written for a scientific audience by John R. Peacock of the University of Edinburgh. The notes are to be regarded as purely supplementary for this course and you will probably profit from looking at, say, pages 1-18, though they take you much further if you choose to go there some day.

2. The Stuff to Remember Forever

a.) Linear Ordinary Differential Equations with *constant* coefficients are special! Their solutions are all exponentials - though possibly with complex arguments which, then, yield our oscillatory behavior.

b.) Any **Sinusoidal** driving force leads to a *Steady State Response* which is also sinusoidal at the exact same driving frequency (though shifted by a constant phase) ! This fact is due solely to the linearity of the defining equation. For the same reason, if the drive is a sum of individual sinusoidal forces, then the steady state response will be the sum of the individual solutions. It's that simple. Linearity is an overwhelmingly important feature.

c.) In the technical person's way of thinking all S.H.O. systems are "self-similar" and may be mapped on to each other by simple scaling. So! ... the first things you ever ask about any S.H.O. is "what are ω_o and Q" ? The System we now consider has one more feature, viz. the drive. Accordingly there are **three** time scales: $\{\, T_o, T_{decay}, T_{drive} \,\}$. This implies that there are **two** *dimensionless ratios*. We take as our standardized expressions of these: $\overline{\omega_d} \;\equiv\; \omega_d/\omega_o$ and $Q \;\equiv\; \omega_o T_{decay}$. Basically *everything* that can be said about the system can be expressed in terms of these.

d.) A driven S.H.O. responds most vigorously if the driving frequency is close to the natural frequency. This is called "Resonance". Just how close you have to get is best described by the transfer of energy discussion. This leads directly to one of the most fundamental *(... and yes, again ... sort of "Bell-Shaped")* functions in all of physics namely the "Lorentzian Power Curve". If you are driving the system at lesser or greater frequencies than the "Natural" frequency ... then your rate of energy transfer drops. If we measure the frequency span from half max power delivery below to half max above we come to an exact relationship worth memorizing: $\Delta\,\overline{\omega}_{1/2} \;=\; 1\,/\,Q$.

3. Our S.H.O. System

Our system obeys the following fundamental relationship:

$$m\,\ddot{x} \;+\; b\,\dot{x} \;+\; k\,x \;\;=\;\; F(t) \tag{1}$$

which we gently rearrange by dividing out the mass and defining: $2\,\beta \equiv b/m$ and $\omega_o^2 \;\equiv\; k/m$ and $f(t) \equiv F(t)/m$. This results in the form of our equation given in our Text:

$$\ddot{x} \;+\; 2\,\beta\,\dot{x} \;+\; \omega_o^2\,x \;\;=\;\; f(t) \tag{2}$$

We put this, however, in a more universal (and far more meaningful) form by defining: $b/m \;=\; 2\,\beta \;\;\equiv\; 1/\,T_{decay}$ and now also, finally, $\omega_o\,T_{decay} \;\equiv\; Q$ to achieve:

$$\ddot{x} \;+\; \frac{\omega_o}{Q}\dot{x} \;+\; \omega_o^2\,x \;\;=\;\; f(t) \qquad \text{our final "working" form for } \textit{Newton's Second Law.} \tag{3}$$

4. Our Fourier Approach

The FORCING FUNCTION for this problem is given by the linear function:

$$f(t)_{\,drive} \;=\; f_{max}\left(\frac{t}{T_d/2}\right) \tag{4}$$

It is very convenient at this point to introduce an *angle* variable θ which will represent the time in a one-to-one way by a simple proportionality. We define θ then via:

$$\frac{\theta}{2\,\pi} \;\equiv\; \frac{t}{T_{drive}} \qquad \text{or} \qquad \theta \;=\; 2\,\pi\,\frac{t}{T_{drive}} \;=\; \omega_d\,t \tag{5}$$

In this way, as t sweeps out an interval of size T_d our angle θ sweeps out a "*normalized*" interval of size $2\,\pi$. Notice that our forcing function can now be easily expressed as:

$$f(t)_{\,drive} \;=\; f_{max}\left(\frac{\theta}{\pi}\right) \tag{6}$$

Now we write the fundamental Fourier statement:

$$f(t)_{\,drive} \;=\; \sum_{n=0}^{\infty}\left(A_n\,cos(n\,\theta) \;+\; B_n\,sin(n\,\theta)\,\right) \tag{7}$$

You will recall that <u>any</u> function may be written as the sum of its even and odd parts. The way to do this is simply to write:

$$f(t) \;=\; \frac{1}{2}\{\,f(t)\,+\,f(-t)\,\} \quad + \quad \frac{1}{2}\{\,f(t)\,-\,f(-t)\,\} \quad \equiv \quad f(t)_{even}\;+\;f(t)_{odd}$$

The cosines (being themselves even) in the Fourier decomposition represent the even portion of our function and the sines (being odd) the remaining odd portion. So, in the case at hand, we only have the sines present.

$$f(t)_{\,drive} \;=\; \sum_{n=1}^{\infty} B_n\,sin(n\,\theta) \tag{8}$$

At this point we use the key *orthogonality* property of our trig functions, namely:

$$\int_{-\pi}^{\pi} sin(m\,\theta)\,sin(n\,\theta)\,d\theta \;=\; \pi\,\delta_{m,n} \tag{9}$$

Using this, we finally conclude that the Fourier coefficients may be extracted by a universal procedure, viz.:

$$\int_{-\pi}^{\pi} sin(m\,\theta)\,f(t)_{\,drive}\,d\theta \;=\; \pi\,B_m \tag{10}$$

Now for the STEADY STATE RESPONSE ! Our system responds with its own Fourier series:

$$x(t)_{\,response} \;=\; \sum_{n=1}^{\infty} x_n\,sin(n\,\theta\,-\,\delta_n) \qquad \ldots \qquad \text{where :} \tag{11}$$

$$x_{\,n} \;=\; \frac{B_n}{\omega_o^2}\,\frac{1}{\sqrt{[1\,-\,(n\,\overline{\omega_d})^2]^2\;+\;[n\,\overline{\omega_d}\,/Q]^2}} \qquad \ldots \qquad \text{and :} \tag{12}$$

$$\tan(\delta_{\,n}) \;=\; \frac{(n\,\overline{\omega_d})\,/Q}{1\,-\,(n\,\overline{\omega_d})^2} \tag{13}$$

C. Your Problems

1. *The Forcing Function*

a.) Please compute analytically the B_n that come from our specified $f(t)_{drive}$. Make a **table** of $\frac{B_n}{f_{max}}$ for the first ten terms and also a **graph** of these numbers versus n.

b.) We now use these terms in our Fourier representation to approximate our true $f(t)_{drive}$:

$$f(t)_{drive} \;\approx\; f(t)_{\,approx} \;\equiv\; \sum_{n=1}^{10} B_n \, sin(n\,\theta) \tag{14}$$

Please **graph** $\frac{f(t)_{approx}}{f_{max}}$ versus θ over the interval $[-\pi\,,\; +\pi\,]$ and superimpose the graph of $\frac{f(t)_{drive}}{f_{max}}$.

2. *The Response Function*

You will now consider four cases which you are to compute and observe the manner of response:

Case 1.) $\overline{\omega_d} \;=\; 1/5$ and $Q \;=\; 3$

Case 2.) $\overline{\omega_d} \;=\; 1/5$ and $Q \;=\; 15$

Case 3.) $\overline{\omega_d} \;=\; 5$ and $Q \;=\; 3$

Case 4.) $\overline{\omega_d} \;=\; 5$ and $Q \;=\; 15$

c.) For each case numerically compute the response coefficients $\{x_n\}$ and form both a **table** and a **graph** of :

$$\frac{x_n}{f_{max}/\omega_o^2} \qquad \text{versus} \qquad n$$

d.) For each case numerically compute the phase shifts $\{\delta_n\}$ and form both a **table** and a **graph** of them versus n.

e.) For each case numerically compute the *approximate response* :

$$x(t)_{response} \;\approx\; x(t)_{\,approx-response} \;\equiv\; \sum_{n=1}^{10} x_n \, sin(n\,\theta \;-\; \delta_n) \tag{15}$$

and make a graph of:

$$\frac{x(t)_{\,approx-response}}{f_{max}/\omega_o^2} \qquad \text{versus} \qquad \theta \qquad \text{over the interval} \qquad [-\pi\,,\,\pi]\,.$$