CSUC
Department of Physics
Class Notes

Informal Notes on "Practical Approximation"

I. *INTRODUCTION*

Differential calculus is, essentially, the study of coordinated small changes. By this we mean, if two quantities are smoothly related, then changing one by a small amount (or "increment" as it is some times expressed) implies that the other will also change by some other corresponding ***small*** amount. Small change on the input side leads to small change on the output side. For example, let's suppose for definiteness that the two quantities are called x and y and are related via the relationship $y = f(x)$. Then, changing x by δx means that y will change by some other small amount which we can call δy. Our job will be to relate these two quantities.

The fundamental observation of differential calculus is that, as these increments become *"small enough"*, then the relationship between them becomes truly *linear* (the straight line relationship) , i.e. that we approach the statement $\delta y \;\propto\; \delta x$. Indeed, if the "smallness" becomes "arbitrarily small" (i.e. "infinitesimal") , then this relation becomes exact. The constant of proportionality we then call the "derivative". We write this in Leibniz notation as:

$$dy \;=\; \frac{dy}{dx}\, dx \tag{1}$$

or also, nowadays, as:

$$dy \;=\; f'(x)\; dx$$

1. *Points of Confusion*

a) The first point of confusion is that equation (1), though *exactly* true only in the limit of arbitrarily small increments $\{dx,\, dy\}$, is often not recognized by young students as still often being an *excellent* approximation even when the increments remain finite (though perhaps "smallish"). In this case we can write:

$$\delta y \;\approx\; f'(x)\, \delta x \tag{2}$$

b) The second point of confusion is a bit more general. In mathematics the term "analysis" encompasses a whole set of closely related topics clustered around the notion of smooth relationship and convergence. Included among these are calculus and the idea of infinite series. So, when we develop practical tools for approximation, we will borrow ideas from different places and they may not seem very related though in the greater scheme of things they really are.

c) The third point is just one of notation. An infinitesimal number is generally indicated by the prefix of a latin "d", i.e. $\{dx,\, dy\}$. The inclusion of the lower case Greek letter delta δ indicates any "smallish" number but not necessarily one that is truly infinitesimal e.g. δx. The use of the upper case Greek letter delta e.g. Δx indicates that the quantity Δx could be *any* number finite *or* infinitesimal.

A. Starting Examples: level 1

Let's start with simple algebra. We know it is *always* true that:

$$(1 + x)^2 = 1 + 2x + x^2 \tag{3}$$

But now suppose, in addition, we happen to know that $x \ll 1$. If it were the case that $x \equiv 0$, then our answer is, of course $(1 + 0)^2 = 1$. But as x pulls away from the value zero our answer starts to change too. We use the simple observation that, for any (positive) nonzero number x, if $x \ll 1$ happens to be true, then this implies $x^2 \ll x$ as well. The largest part of the change in our output value is consequently going to be the $2x$ term. This allows us to do some very useful " quick arithmetic "! For example:

$$(1 + .01)^2 \approx 1 + .02 + \textit{h.p.t.}$$

The designation "h.p..t." stands for "higher power terms" and in this case includes a correction one hundred times smaller than the correction we kept. Since we know the Binomial Theorem for *any* power, we can immediately generalize our result to:

$$(1 + x)^n = 1 + nx + \textit{h.p.t.} \tag{4}$$

Actually, this result may *also* be understood as just an example of equation (2) if the function is $(1 + x)^n$ and we start from $x = 0$. It's true for *any* power n , even fractional powers as long as $x \ll 1$! Consider for example:

$$\begin{aligned} (1 + x)^{\frac{1}{2}} &\approx 1 + \frac{1}{2}x + \textit{h.p.t.} \\ (1 + x)^{-\frac{1}{2}} &\approx 1 - \frac{1}{2}x + \textit{h.p.t.} \end{aligned}$$

An example of particular interest in what is to follow is:

$$(1 + x)^{-\frac{3}{2}} \approx 1 - \frac{3}{2}x + \textit{h.p.t.} \tag{5}$$

B. Starting Examples: level 2

We will often need to know, too, how functions of *vectors* change as the input vector changes by a small amount. This sounds complicated but actually won't be too bad. We extract numbers out of vectors by taking *dot products*. Recall that the magnitude $R \equiv |\vec{R}|$ of the vector $\vec{R}$ is found via dot products as:

$$R^2 = \vec{R} \cdot \vec{R} \tag{6}$$

Suppose now we add a "small" vector, let's call it $\vec{s}$, to $\vec{R}$ and ask by how much the magnitude of the sum will change from what it was before adding $\vec{s}$. That is, if $|\vec{s}| \ll |\vec{R}|$, how does $|\vec{R} + \vec{s}|$ depend on $\vec{s}$ to "first order" in the "small" quantity $|\vec{s}|$? A simple way to start is to write:

$$|\vec{R} + \vec{s}|^2 \equiv (\vec{R} + \vec{s}) \cdot (\vec{R} + \vec{s}) = \vec{R} \cdot \vec{R} + 2\vec{R} \cdot \vec{s} + \vec{s} \cdot \vec{s} \tag{7}$$

So now, if we drop the term $\vec{s} \cdot \vec{s}$ as being of size "small squared" (and thus negligible in out approximation), we achieve:

$$|\vec{R} + \vec{s}|^2 \approx R^2 + 2\vec{R} \cdot \vec{s} + \textit{h.p.t.} = R^2 \left(1 + 2\frac{\vec{R}}{R} \cdot \frac{\vec{s}}{R}\right) \tag{8}$$

Then, if we use the handy unit vector notation $\hat{R} \equiv \vec{R}/R$, we can write:

$$|\vec{R} + \vec{s}|^2 \approx R^2 \left(1 + 2\hat{R} \cdot \frac{\vec{s}}{R}\right) \tag{9}$$

And finally, if we take the square root of this equation and use what we learned above, we arrive at:

$$|\,\vec{R} + \vec{s}\,| \approx R\left(1 + \hat{R}\cdot\frac{\vec{s}}{R}\right) \tag{10}$$

This equation can be further used to generate even more good approximations! For example:

$$\frac{1}{|\,\vec{R} + \vec{s}\,|^{3}} = |\,\vec{R} + \vec{s}\,|^{-3} \approx R^{-3}\left(1 - 3\,\hat{R}\cdot\frac{\vec{s}}{R}\right) \tag{11}$$

This result appears in the classic "*dipole*" approximation derived next. We start by considering:

$$\frac{(\vec{R} + \vec{s})}{|\,\vec{R} + \vec{s}\,|^{3}} \approx (\vec{R} + \vec{s})\,R^{-3}\left(1 - 3\,\hat{R}\cdot\frac{\vec{s}}{R}\right) \tag{12}$$

Now we collect and keep only those terms no smaller than "first order" (i.e. single power) in $\vec{s}$.

$$\frac{(\vec{R} + \vec{s})}{|\,\vec{R} + \vec{s}\,|^{3}} \approx R^{-3}\left(\vec{R} + \vec{s} - 3\,\vec{R}\,\hat{R}\cdot\frac{\vec{s}}{R}\right) \tag{13}$$

A final tidying up (using our unit vector notation) yields:

$$\frac{(\vec{R} + \vec{s})}{|\,\vec{R} + \vec{s}\,|^{3}} \approx R^{-2}\left(\hat{R} + \frac{\vec{s}}{R} - 3\,\hat{R}\,\hat{R}\cdot\frac{\vec{s}}{R}\right) \tag{14}$$

C. Summary

We may now conclude with a convenient summary of our "*dipole*" approximation in conventional notation. Since, by definition we have:

$$\frac{\vec{R}}{|\,\vec{R}\,|^{3}} \equiv \frac{\hat{R}}{R^{2}} \tag{15}$$

then, as we add a small vector $\vec{s}$ to $\vec{R}$, we approximately have:

$$\frac{(\vec{R} + \vec{s})}{|\,\vec{R} + \vec{s}\,|^{3}} \approx \frac{\hat{R}}{R^{2}} + \frac{1}{R^{2}}\left(\frac{\vec{s}}{R} - 3\,\hat{R}\,\frac{\hat{R}\cdot\vec{s}}{R}\right) \tag{16}$$