CSUC
Department of Physics

I. *ELLIPSES AND THEIR PROPERTIES*

Ellipses, hyperbolas and parabolas were known, historically, more for how they were *drawn* in a plane rather than for being e.g. specifically *conic sections*. In general, they all emerge from the same standard picture. We start by drawing a point called the *focus* and a line called the *directrix*. The curve consists then of all those points w in the plane such that the distance $\mathbf{r}$ from the focus to w divided by the perpendicular distance $\mathbf{d}$ from w to the directrix forms a constant ratio traditionally called the "eccentricity" ϵ , i.e. all points w such that :

$$\frac{\mathbf{r}}{\mathbf{d}} = \epsilon \tag{1}$$

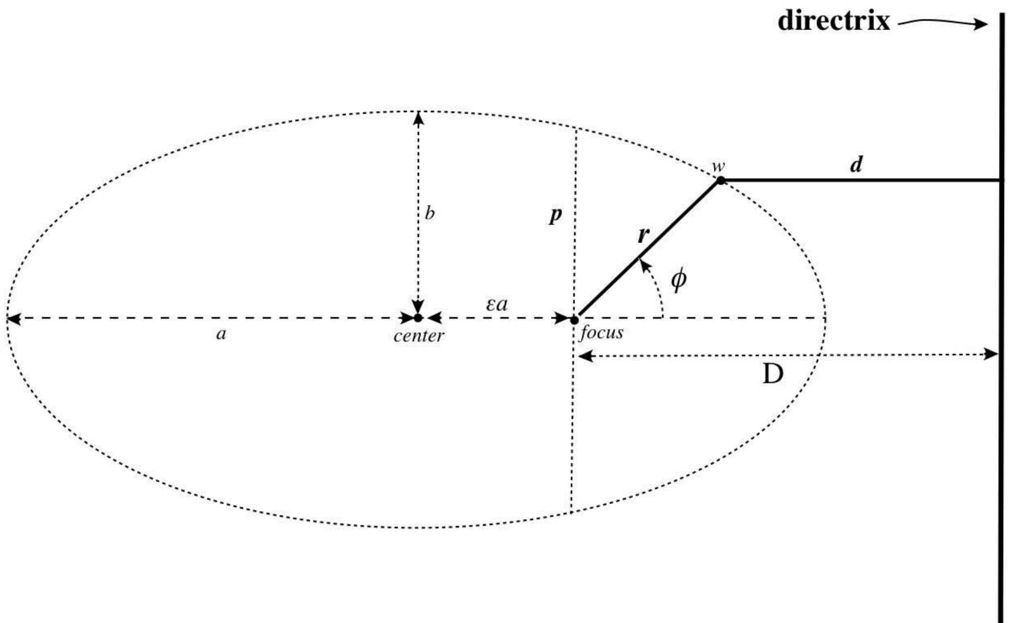

FIG. 1: The geometry of the Conic Sections.

The three cases are generated by specific choices for ϵ :

$$\begin{aligned} \epsilon\ &> 1 \ \mapsto hyperbola \\ \epsilon\ &= 1 \ \mapsto parabola \\ \epsilon\ &< 1 \ \mapsto ellipse \end{aligned}$$

Notice that $\mathbf{d} = D - \mathbf{r}\cos(\phi)$ so ...

$$\frac{\mathbf{r}}{D - \mathbf{r}\cos(\phi)} = \epsilon \tag{2}$$

This we arrange into the standard form:

$$\frac{\epsilon D}{\mathbf{r}} = 1 + \epsilon \cos(\phi) \tag{3}$$

Following tradition, we write $\epsilon D \equiv \mathbf{p}$ and we then have as our general equation:

$$\frac{\mathbf{p}}{\mathbf{r}} = 1 + \epsilon \cos(\phi) \tag{4}$$

Notice especially that $\mathbf{r} = \mathbf{p}$ corresponds to an angle of $\phi = \frac{\pi}{2}$ that is, at right angles to the major axis of the drawing and for this reason $\mathbf{p}$ is known as the **semi-latus-rectum** . It will play an especially central role in these figures and serves as the natural unit of length.

The following further features may be related to $\mathbf{p}$ and ϵ via simple analytic geometry. we arrive at:

$$\begin{aligned}
semi-major-axis:\quad \mathbf{a}/\mathbf{p} &= \frac{1}{1-\epsilon^2} \\
semi-minor-axis:\quad \mathbf{b}/\mathbf{p} &= \frac{1}{\sqrt{1-\epsilon^2}} \\
\mathbf{r}_{min} = \frac{\mathbf{p}}{1+\epsilon} &= \mathbf{a}\,(1-\epsilon) \\
\mathbf{r}_{max} = \frac{\mathbf{p}}{1-\epsilon} &= \mathbf{a}\,(1+\epsilon)
\end{aligned}$$

II. KEPLER'S CONSTRUCTION

Notice especially that this earlier and more encompassing description of the conic sections described above {ellipse, hyperbola, parabola} is very "*polar coordinate-like*". These days students are far more likely to learn a Cartesian expression instead. Actually, there *is* something more to learn from the Cartesian approach! We present, next, a famous geometric construction that bears Kepler's name. A second understanding of what an ellipse "*is*" . . . comes from realizing that an ellipse truly ***is*** a circle that has been "squashed" in one direction. In fact, in the next figure we see together the original "reference circle" and the ellipse it has been "squashed" into by reducing all the vertical ("y") coordinates by the same constant factor = $\mathbf{b}/\mathbf{a}$.

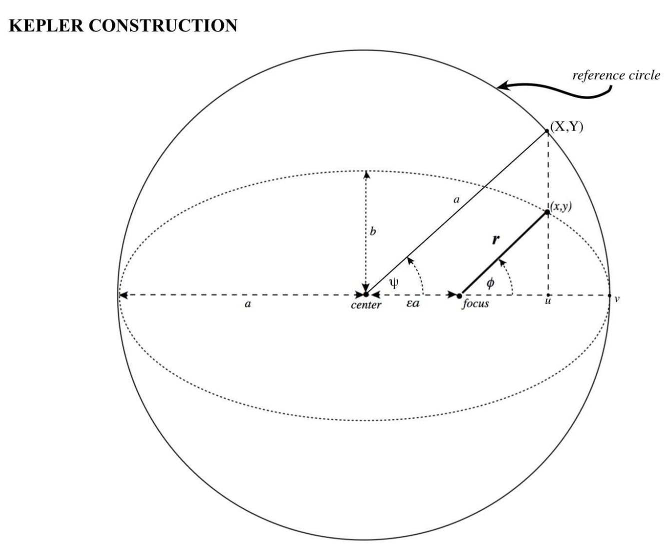

FIG. 2: An ellipse and its reference circle.

Every point on the reference circle looks straight down on the point it matches to on the ellipse below. In the picture shown, the coordinate "Y" on the circle gets shrunk down to the coordinate "y" on the ellipse: i.e. $y = \frac{\mathbf{b}}{\mathbf{a}} Y$. The equation of the circle when we use its center as our origin is, of course:

$$(\frac{X}{\mathbf{a}})^2 + (\frac{Y}{\mathbf{a}})^2 \; = \; 1 \tag{5}$$

And the well known equation of the ellipse is then just like it, only scaling the horizontal and vertical dimensions differently:

$$(\frac{x}{\mathbf{a}})^2 + (\frac{y}{\mathbf{b}})^2 \; = \; 1 \tag{6}$$

The solution may be parameterized as:

$$x \; = \; \mathbf{a} \, \cos(\psi) \tag{7}$$

$$y \; = \; \mathbf{b} \, \sin(\psi) \tag{8}$$

Kepler realized that since only the vertical dimension has been reduced by this constant factor, that the area of the ellipse itself must have been shrunk by just the same factor. We may write $(Area)_{ellipse} = \frac{\mathbf{b}}{\mathbf{a}}(Area)_{circle}$. This leads directly to our first result:

$$(Area)_{ellipse} = \frac{\mathbf{b}}{\mathbf{a}} \, \pi \, \mathbf{a}^2 \; = \; \pi \, \mathbf{a} \, \mathbf{b} \tag{9}$$

However, and here's the genius, Kepler further realized that the same result must also hold for corresponding *sectors* too ! The curved portion of the ellipse bounded by *y-u-v* must be just a scaled down version of the curved portion of the circle bounded by *Y-u-v* . Indeed, we may write:

$$(Area)_{y\text{-}u\text{-}v} = \frac{\mathbf{b}}{\mathbf{a}}(Area)_{Y\text{-}u\text{-}v} \tag{10}$$

In the end, this is how Kepler effected what we would now call a tricky integration ... but without the aid of calculus ! Kepler's second law requires knowing the area of the sector *(y-focus-v)*. Since the area of the circular sector *(Y-center-v)* is known to be:

$$(Area)_{(Y\text{-}center\text{-}v)} = \frac{\psi}{2\,\pi} \, \pi \mathbf{a}^2 \tag{11}$$

Now we subtract the triangle $\triangle$*(Y-center-u)* and achieve:

$$\frac{\psi}{2\,\pi} \, \pi \mathbf{a}^2 - \frac{1}{2}(\mathbf{a}\cos(\psi)) \, (\mathbf{a}\sin(\psi)) = (Area)_{Y\text{-}u\text{-}v} = \frac{\mathbf{a}}{\mathbf{b}}(Area)_{y\text{-}u\text{-}v} \tag{12}$$

so that

$$(Area)_{y\text{-}u\text{-}v} = \frac{1}{2}\mathbf{a}\,\mathbf{b}(\psi - \sin(\psi) \, \cos(\psi)) \tag{13}$$

Finally we add in the triangle $\triangle$*(y-focus-u)* and achieve:

$$(Area)_{y\text{-}focus\text{-}v} = \frac{1}{2}\mathbf{a}\,\mathbf{b}(\psi - \epsilon \, \sin(\psi)) \tag{14}$$

This is the desired result! which may be related back to the angle ϕ by a straightforward (but strenuous) piece of algebra yielding:

$$\tan(\frac{\psi}{2}) \; = \; \sqrt{\frac{1-\epsilon}{1+\epsilon}} \, \tan(\frac{\phi}{2}) \tag{15}$$

These last two equations allow us to find the area of a sector of the ellipse in terms of the (physical) polar angle ϕ .