Physics 301A Notes: "Evolute & Involute"

Themes:

1) Vector notation offers a coordinate-free discussion.

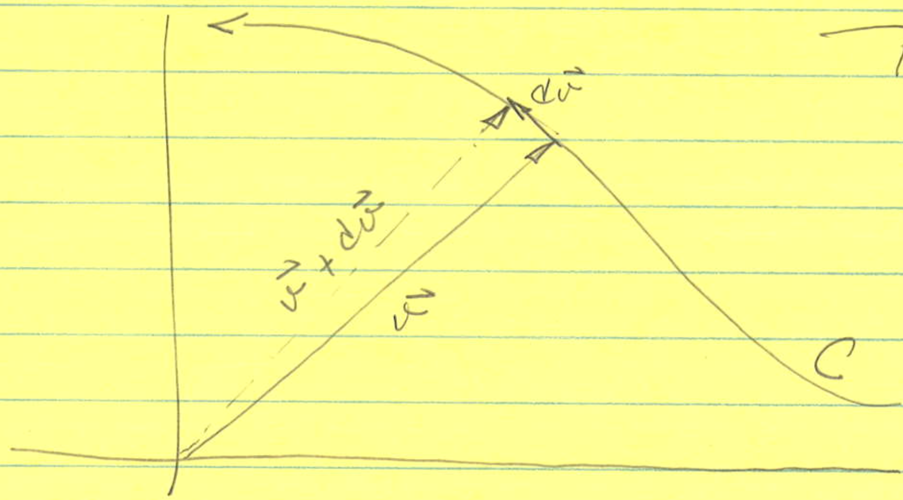
2) Differential notation offers a highly intuitive introduction to the elementary notions of differential geometry.

Motivation: We are preparing for Lagrangian-Mechanics. We will need to distinguish between the features of the problem which are purely "Geometric", & those that are essentially "Dynamic".

Prologue: This initial study is restricted to two dimensional (2-D) plane geometry. Non-the-less, we will exploit the 3rd dimension (I will call it $\hat{z}$) to simplify algebra by using cross-products! (this is "gentle cheating").

Our center of attention will be those smooth curves in our plane that have, at most, a finite number of discontinuities, & "sharp points". Sometimes we will find it convenient (but never necessary) to "parametrize" our curves, e.g. $\vec{r}(t)$.

Our Setting:



Take a smooth curve C:

A small displacement $d\vec{v}$ along our curve from $\vec{v}$ to $\vec{v}+d\vec{v}$ defines both a tangent direction $\hat{\tau}$ and an element of length $ds \equiv |d\vec{v}|$.

i.e. $d\vec{v} = ds\,\hat{\tau}$ where $\hat{\tau} \equiv \dfrac{d\vec{v}}{|d\vec{v}|} = \dfrac{d\vec{v}}{ds}$

Example: Suppose our curve is the parabola $y = \dfrac{x^2}{2}$.

Then $\vec{v} = \langle x, y \rangle = \langle x, \frac{x^2}{2} \rangle$

In this notation x serves as "parameter" and coordinate:

So $d\vec{v} = \langle dx, dy \rangle = \langle dx, x\,dx \rangle = \langle 1, x \rangle dx$

$\Longrightarrow ds = |d\vec{v}| = \sqrt{dx^2 + dy^2} = \sqrt{1+x^2}\,dx$

$\hat{\tau} = \dfrac{d\vec{v}}{ds} = \dfrac{\langle 1, x \rangle dx}{\sqrt{1+x^2}\,dx} = \dfrac{\langle 1, x \rangle}{\sqrt{1+x^2}}$

Now add in time to our basic discussion

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{ds}{dt}\frac{d\vec{r}}{ds} = v\,\hat{\tau}$$

kinematics geometry.

Since we also need acceleration, we must study $d\vec{v}$.

$$d\vec{v} = dv\,\hat{\tau} + v\,d\hat{\tau}$$

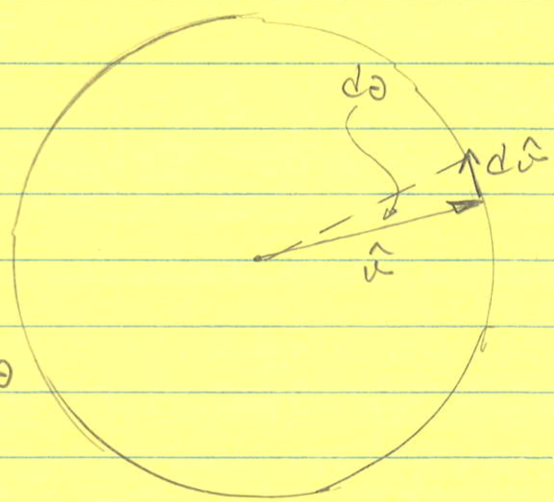
Observe! The new thing is "changes in unit vectors".

In particular: 1) Any differential change in a unit vector is perpendicular to the vector itself.
2) Any differential change in a unit vector defines an angle

So: $\hat{v}\cdot\hat{v} \equiv 1$

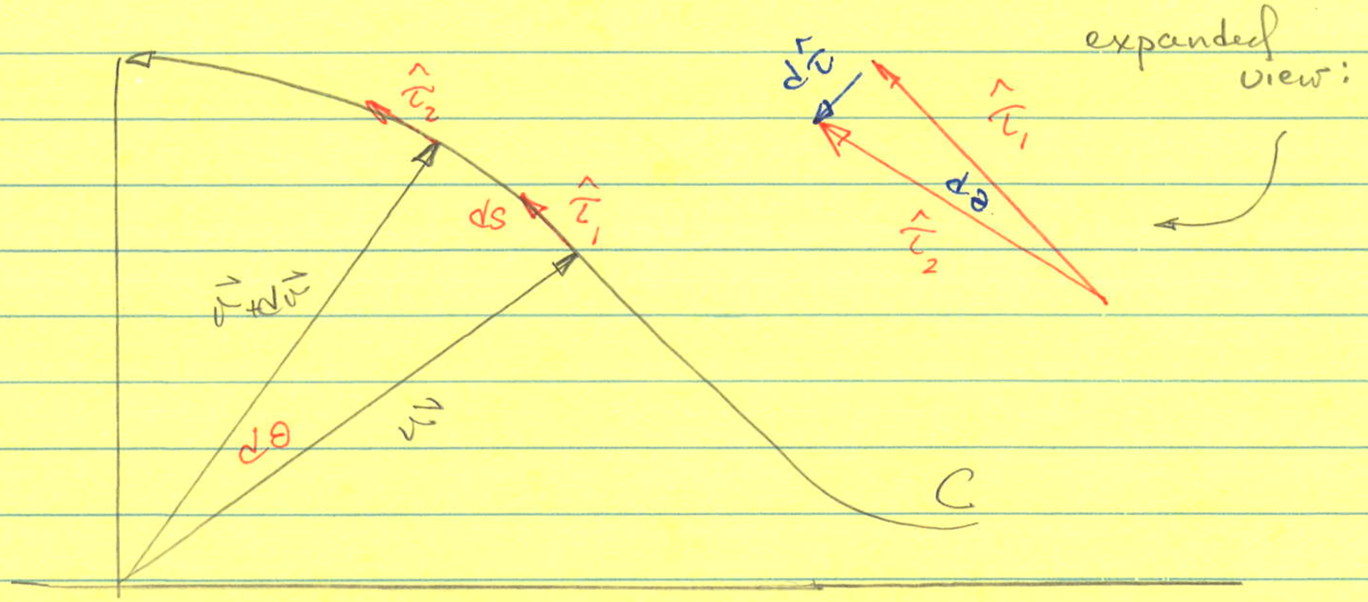
$\Longrightarrow \hat{v}\cdot d\hat{v} = 0$

& $|d\hat{v}| = |\hat{v}|\,d\theta = d\theta$



In the case of our tangent direction $\hat{\tau}$,
$d\hat{\tau} = \hat{n}\,d\theta$ where $\hat{n} = \hat{z}\times\hat{\tau}$

So! $d\vec{v} = dv\,\hat{\tau} + v\,d\theta\,\hat{n}$



As we proceed along our curve from $\vec{v}$ to $\vec{v}+d\vec{v}$, the tangent direction at $\vec{v}$, $\hat{\tau}_1$, proceeds to $\hat{\tau}_2 = \hat{\tau}_1 + d\hat{\tau}$

In this travel we have passed through a distance ds and our tangent has rotated by $d\theta$. These quantities define a natural radius of length ρ where

$$\rho\, d\theta = ds \quad \text{or} \quad \rho = \frac{ds}{d\theta} \quad \text{or} \quad \frac{1}{\rho} = \frac{d\theta}{ds}.$$

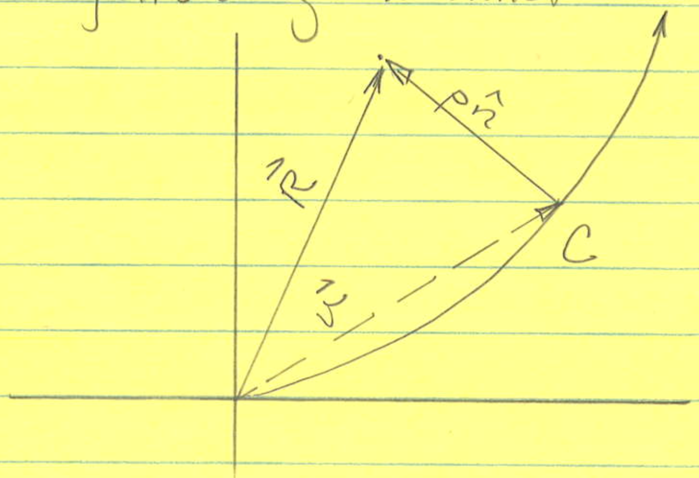
So we may write, quite naturally: $d\theta = \dfrac{d\theta}{ds}ds = \dfrac{ds}{\rho}$

$$\Longrightarrow \quad d\vec{v} = dv\,\hat{\tau} + v\frac{ds}{\rho}\,\hat{n} \quad \text{and, now,}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{dv}{dt}\hat{\tau} + \frac{v^2}{\rho}\hat{n} \quad \text{just like a circle!}$$

The Evolute:

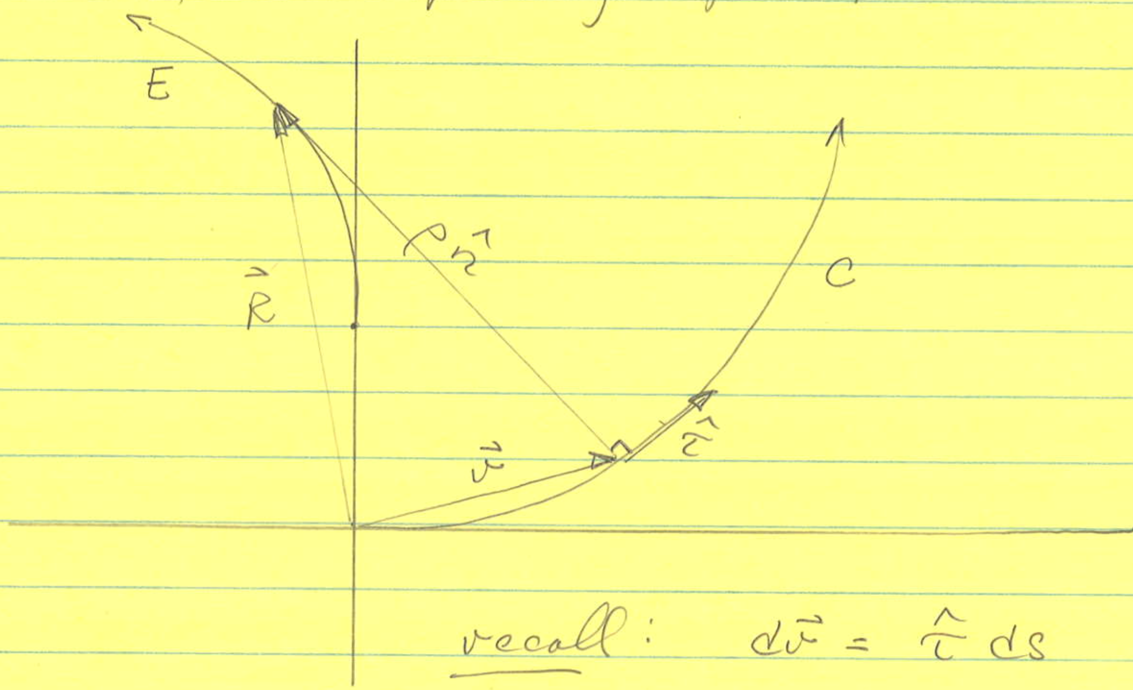
For any smooth curve C we may define a "sibling" curve E, the evolute, point for point in the following manner



If $\vec{v}$ points to a spot on C, let $\vec{R}$ point to the corresponding "center of curvature" spot.

$$\vec{R} \equiv \vec{v} + \rho\hat{n}$$

The curve E is the collection of all the centers of curvature — point for point!



recall: $d\vec{v} = \hat{\tau}\, ds$

$d\hat{\tau} = \hat{n}\, d\theta$

where $\hat{n} = \hat{z} \times \hat{\tau}$; $d\hat{n} = -\hat{\tau}\, d\theta$.

This last result follows from our vector algebra, as follows:

If $\hat{n} = \hat{z} \times \hat{\tau}$ then $d\hat{n} = \hat{z} \times d\hat{\tau} = \hat{z} \times (\hat{n}\, d\theta)$

and so, $d\hat{n} = \hat{z} \times (\hat{z} \times \hat{\tau}\, d\theta)$, use double cross product!

$$= \left\{ \hat{z}(\hat{z} \cdot \hat{\tau}) - \hat{\tau}(\hat{z} \cdot \hat{z}) \right\} d\theta = -\hat{\tau}\, d\theta$$

We may also observe, if $d\hat{\tau} = \hat{n}\, d\theta$ then $\hat{n} \cdot d\hat{\tau} = d\theta = \frac{ds}{\rho}$

or $\frac{1}{\rho} = \hat{n} \cdot \frac{d\hat{\tau}}{ds} = (\hat{z} \times \hat{\tau}) \cdot \frac{d\hat{\tau}}{ds}$ (interchange dot, & cross ...)

and finally: $\frac{1}{\rho} = \hat{z} \cdot \left(\hat{\tau} \times \frac{d\hat{\tau}}{ds} \right)$ (whew!)

Before returning to our evolute, let's collect our results!

a) $d\vec{r} = \hat{\tau}\, ds$

b) $d\hat{\tau} = \hat{n}\, d\theta$

c) $d\hat{n} = -\hat{\tau}\, d\theta$

where, then, $\hat{n} = \hat{z} \times \hat{\tau}$, $ds = \rho\, d\theta$ and, finally

d) $\frac{1}{\rho} = \hat{z} \cdot \left(\hat{\tau} \times \frac{d\hat{\tau}}{ds} \right)$

Back to the evolute! Start from curve C,

As $\vec{r}$ marks out C, then point for point

$\vec{R}$ marks out E via the prescription

$\vec{R} \equiv \vec{r} + \rho\hat{n}$. Now observe!

$$d\vec{R} = d\vec{r} + d\rho\,\hat{n} + \rho\, d\hat{n}$$

<u>but</u> $d\vec{r} = ds\,\hat{\tau}$ & $\rho\, d\hat{n} = -\rho\,\hat{\tau}\, d\theta = -ds\,\hat{\tau}$

these terms <u>cancel</u>, so $d\vec{R} = d\rho\,\hat{n}$. What does it mean?

Since $d\vec{R}$ is <u>tangent</u> to E at $\vec{R}$ & $|d\vec{R}| = dS$ ...,

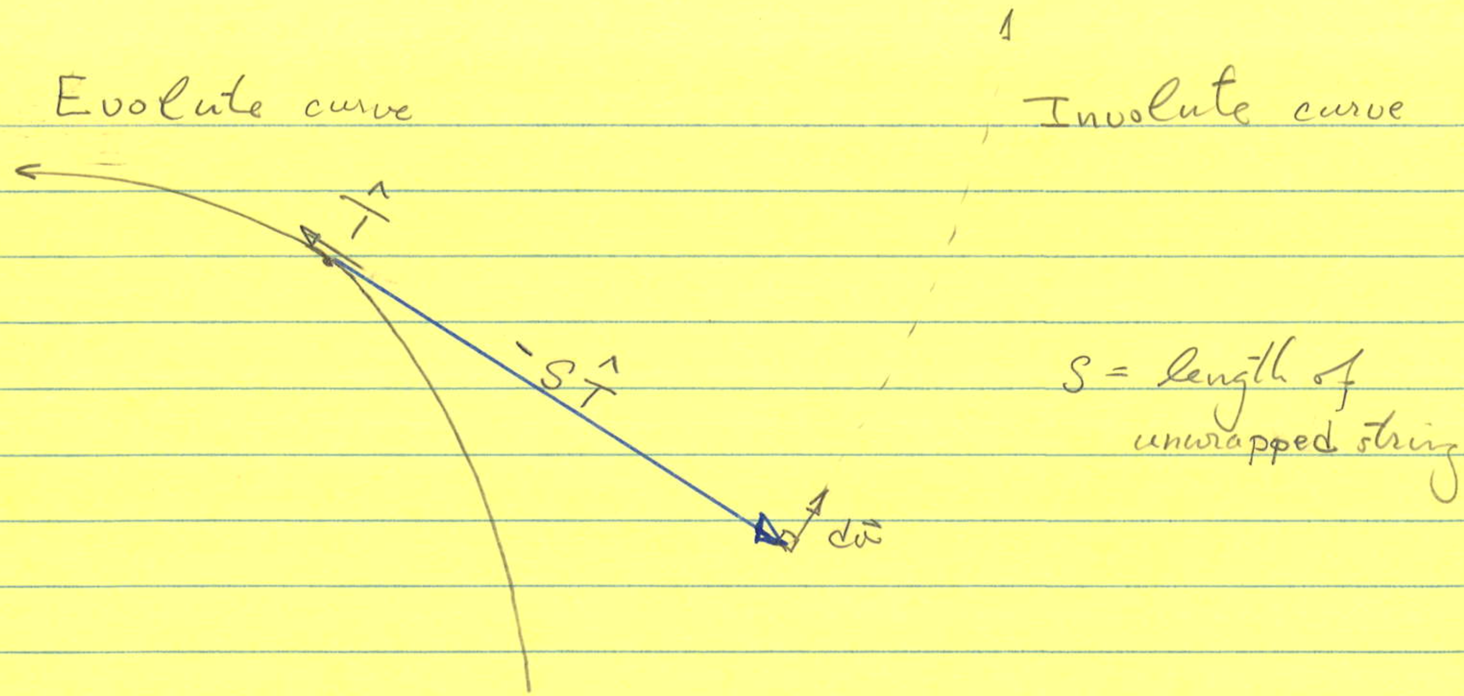
apparently $|d\vec{R}| = \pm d\rho$ ...

and the <u>normal</u> to C is the <u>tangent</u> to E.

Think what this means. If we had <u>started</u> with E, then its <u>tangent</u> $\hat{T}$ points along $\hat{n}$. But a string "unwinding" from E would have exactly this property. We could have defined $\vec{r}$ from $\vec{R}$ by

$$\vec{r} = \vec{R} - S\hat{T} \implies d\vec{r} = d\vec{R} - dS\,\hat{T} - S\, d\hat{T}$$

but then $d\vec{R} = dS\,\hat{T}$ so $d\vec{r} = -S\, d\hat{T} = -S\, d\theta\, \hat{N}$.



By "unwinding" a string from the evolute curve we create the involute curve.

By finding the centers of curvature of the involute curve, we find the evolute curve.

Lets finish this discussion by expressing our results as they would appear had we used an arbitrary parameterization ... and then find the evolute of a simple parabola!

We assume, then, that we are given $\vec{r}(t)$ where "t" is <u>any</u> chosen parameter.

$$d\vec{r} = \langle dx, dy \rangle = \langle \dot{x}, \dot{y} \rangle \, dt$$

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{\dot{x}^2 + \dot{y}^2} \, dt$$

$$\hat{\tau} = \frac{d\vec{r}}{ds} = \frac{\dot{\vec{r}}}{\dot{s}} = \frac{\langle \dot{x}, \dot{y} \rangle}{\sqrt{\dot{x}^2 + \dot{y}^2}}$$

$$\hat{n} = \hat{z} \times \hat{\tau} = \frac{\langle -\dot{y}, \dot{x} \rangle}{\sqrt{\dot{x}^2 + \dot{y}^2}}$$

$$\frac{1}{\rho} = \hat{n} \cdot \frac{d\hat{\tau}}{ds} = (\hat{z} \times \hat{\tau}) \cdot \frac{\dot{\hat{\tau}}}{\dot{s}}$$

but $\dot{\hat{\tau}} = \frac{d}{dt}\left(\frac{\dot{\vec{r}}}{\dot{s}}\right) = \left(\frac{\ddot{\vec{r}}}{\dot{s}} - \frac{\dot{\vec{r}}}{\dot{s}^2}\ddot{s}\right)$

so! $\frac{1}{\rho} = \left(\hat{z} \times \frac{\dot{\vec{r}}}{\dot{s}}\right) \cdot \left\{ \frac{\ddot{\vec{r}}}{\dot{s}} - \frac{\dot{\vec{r}}}{\dot{s}^2}\ddot{s} \right\} \frac{1}{\dot{s}}$

since $(\hat{z} \times \dot{\vec{r}}) \cdot \dot{\vec{r}} \equiv 0$ ... we get...

$$\frac{1}{\rho} = \frac{(\hat{z} \times \dot{\vec{r}}) \cdot \ddot{\vec{r}}}{\dot{s}^3} = \frac{\hat{z} \cdot (\dot{\vec{r}} \times \ddot{\vec{r}})}{\dot{s}^3}$$ or ...

$$\frac{1}{\rho} = \frac{\ddot{y}\dot{x} - \ddot{x}\dot{y}}{(\dot{x}^2 + \dot{y}^2)^{3/2}} \quad !$$

Now apply all this to our parabola!

If we start with $y = \frac{1}{2}x^2$, then

$\vec{r} = \langle x, y \rangle = \langle x, \frac{1}{2}x^2 \rangle$.

Let our parameter "t" be x.

So: $\dot{x} = 1$, $\ddot{x} = 0$

$ds = \sqrt{1+x^2}\, dx$

$\dot{y} = x$, $\ddot{y} = 1$

then $$\frac{1}{\rho} = \frac{\ddot{y}\dot{x} - \ddot{x}\dot{y}}{(\dot{x}^2+\dot{y}^2)^{3/2}} = \frac{1}{(1+x^2)^{3/2}}$$

so $\rho = (1+x^2)^{3/2}$

$$\hat{\tau} = \frac{\langle 1, x \rangle}{\sqrt{1+x^2}} \qquad \hat{n} = \frac{\langle -x, 1 \rangle}{\sqrt{1+x^2}}$$

$$\vec{R} = \vec{r} + \rho\hat{n} = \langle x, \frac{x^2}{2} \rangle + (1+x^2)^{3/2} \frac{\langle -x, 1 \rangle}{\sqrt{1+x^2}}$$

$$\vec{R} = \langle -x^3, 1 + \frac{3}{2}x^2 \rangle$$

So, if $\vec{R} = \langle \underline{X}, \underline{Y} \rangle$, then

$$\left.\begin{aligned} \underline{X} &= -x^3 \\ \underline{Y} &= 1 + \tfrac{3}{2}x^2 \end{aligned}\right\}$$ is our parameterization of our evolute!

The slope of the evolute is

$$\frac{d\underline{Y}}{d\underline{X}} = -\frac{1}{x}$$