CSUC
Department of Physics
301A Mechanics

I. *INTEGRAL THEOREMS*

A. Introduction

The integral theorems of mathematical physics all have their origin in the ordinary fundamental theorem of calculus, i.e.

$$\int_{x_a}^{x_b} \frac{df}{dx} dx = f(x_b) - f(x_a) \tag{1}$$

Using this theorem multiple times we may generalize to 2 and 3 dimensional geometries (or even beyond!). In all cases we find the same general pattern viz. : " The Integral of a derivative yields the "*function itself* " summed up over all the boundary points." Algebra in multiple dimensions is facilitated by using vector notation such as $\vec{r}(t)$. So also, a function $\mathbf{F}(x, y, z)$ may be written $\mathbf{F}(\vec{r})$. The ordinary rules of partial differentiation govern our calculus e.g.

$$\frac{d}{dt}F(\vec{r}(t)) = \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} \tag{2}$$

Here also, a judicious choice of notation helps us condense our expressions. In particular, the *gradient* notation $\vec{\nabla}$ will be of particular use in writing our expressions concisely.

$$\vec{\nabla} \equiv \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z} \tag{3}$$

note that the $\hat{\mathbf{x}}$ and the other corresponding terms represent unit vectors in the corresponding directions. The definition expressed in equation (3) gives us a notational tool that allows us to martial our three basic partial derivatives efficiently. In particular, we may write in handy compact notation:

$$\frac{dF}{dt} = \vec{\nabla}F \cdot \frac{d\vec{\mathbf{r}}}{dt} \tag{4}$$

B. The fundamental theorem for line integrals

Assertion:

$$\int_{\vec{r}_i}^{\vec{r}_f} d\tilde{\mathbf{r}} \cdot \vec{\nabla}F = F(\tilde{\mathbf{r}}_f) - F(\tilde{\mathbf{r}}_i) \tag{5}$$

Proof: Ask yourself ... how would you actually perform the integral? The answer is that you would supply some parametrization of the curve 'C' i.e. $\tilde{\mathbf{r}}(s)$ where $\tilde{\mathbf{r}}(s_i) = \tilde{\mathbf{r}}_i$ and $\tilde{\mathbf{r}}(s_f) = \tilde{\mathbf{r}}_f$ and $\tilde{\mathbf{r}}(s)$ traces out the curve C as $s \subset [s_i, s_f]$ proceeds to sweep through *its* values. Then

$$dF \;=\; ds\,\frac{dF}{ds} \;=\; ds\left(\frac{d\vec{\mathbf{r}}}{ds}\cdot\vec{\nabla}F\right) \;=\; \left(ds\,\frac{d\vec{\mathbf{r}}}{ds}\right)\cdot\vec{\nabla}F \;=\; d\tilde{\mathbf{r}}\cdot\vec{\nabla}F \tag{6}$$

So

$$\int_{\vec{r}_i}^{\vec{r}_f} d\tilde{\mathbf{r}}\cdot\vec{\nabla}F \;=\; \int_{s_i}^{s_f} ds\frac{dF}{ds} \;=\; \int_{\tilde{\mathbf{r}}_i}^{\tilde{\mathbf{r}}_f} dF \;=\; F(\tilde{\mathbf{r}}_f) - F(\tilde{\mathbf{r}}_i) \tag{7}$$

The pattern is universal. In each case we consider an integral and then ask how we would actually perform it. A simple parametrization leads immediately to the theorem.

C. Green's Theorem in the Plane

Suppose we have a function $Q(x, y)$ and a region S_{xy} in the $x - y$ plane bounded by a curve C . Suppose further that we must evaluate

$$\iint_{S_{xy}} dx\, dy\, \frac{\partial Q}{\partial y}$$

Consider the above ... how would you actually *do* it? In general, double integrals are evaluated as iterated ordinary one-dimensional integrals.

$$\begin{aligned}
\iint_{S_{xy}} dx\, dy\, \frac{\partial Q}{\partial y} &= \int_a^b dx \int_{Y_1(x)}^{Y_2(x)} \frac{\partial Q}{\partial y} \\
&= \int_a^b dx\{Q(x, Y_2(x)) - Q(x, Y_1(x))\} \\
&= \int_a^b dx\, Q(x, Y_2(x)) + \int_b^a dx\, Q(x, Y_1(x))
\end{aligned}$$

From Equation 8 that this is just what we *mean* by the line integral:

$$-\oint_C Q(x, y)\, dx$$

Implicitly here, we traverse the boundary curve C in a counter-clockwise (positive) manner unless otherwise noted.

In summary then:

$$\iint_{S_{xy}} dx\, dy\, \frac{\partial Q}{\partial y} = -\oint_C Q\, dx \quad \textit{and} \quad \iint_{S_{xy}} dx\, dy\, \frac{\partial P}{\partial x} = \oint_C P\, dy$$

Finally, we note that by adding these two results we can configure the resulting identity in a very suggestive form:

$$\iint_{S_{xy}} dx\, dy \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) = \oint_C (A_x\, dx + A_y\, dy)$$

D. Stokes Theorem

A surface S bounded by a curve C. Consider

$$\oint_C A(x, y, z)\, dx$$

How would we do it?

$$\oint_C A(x, y, z)\, dx = \oint_{C^*} A(x, y, z(x, y))\, dx = \oint_{C^*} \Phi(x, y)\, dx$$

If

$$\Phi(x, y) = A(x, y, z(x, y)).$$

Now

$$\oint_{C^*} \Phi(x, y)\, dx = -\iint_{S_{xy}} dxdy\, \frac{\partial \Phi}{\partial y}$$

and

$$\frac{\partial \Phi}{\partial y} = \frac{\partial A}{\partial y} + \frac{\partial A}{\partial z}\,\frac{\partial z}{\partial y}$$

Notice! The surface normal to S at point (x, y, z) is parallel to

$$\nabla(z - z(x,y)) = (-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1) \;\; \| \;\; \hat{n}.$$

So

$$\frac{\partial z}{\partial y} = -\frac{n_y}{n_z}$$

Recall, that

$$d\sigma = \frac{d\sigma_{xy}}{n_z}$$

So!

$$\begin{aligned}
\oint A\,dx &= -\iint_{S_{xy}} dxdy \left(\frac{\partial A}{\partial y} + \frac{\partial A}{\partial z}\,\frac{\partial z}{\partial y} \right) \\
&= -\iint_{S_{xy}} \frac{dx\,dy}{n_z}\, n_z \left(\frac{\partial A}{\partial y} - \frac{\partial A}{\partial z}\,\frac{n_y}{n_z} \right) \\
&= \iint_S d\sigma \left(\frac{\partial A}{\partial z} n_y - \frac{\partial A}{\partial y}\, n_z \right)
\end{aligned}$$

Now, by adding in the equivalent terms from y and z components, we achieve:

$$\begin{aligned}
&\oint_C (A_x\,dx + A_y\,dy + A_z\,dz) \\
&= \iint_S d\sigma \left(\left(n_y \frac{\partial}{\partial z} A_x - n_z \frac{\partial}{\partial y} A_x \right) + \left(n_z \frac{\partial}{\partial x} A_y - n_x \frac{\partial}{\partial z} A_y \right) + \left(n_x \frac{\partial}{\partial y} A_z - n_y \frac{\partial}{\partial x} A_z \right) \right) \\
&= \iint_S d\sigma \left(\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) n_x + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) n_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) n_z \right)
\end{aligned}$$

In modern notation this appears substantially condensed as:

$$\oint \vec{A} \cdot d\vec{r} = \iint_S \vec{\nabla} \times \vec{A} \cdot \hat{n}\, d\sigma$$

E. Divergence Theorem

For a given function $R(x, y, z)$, consider

$$\iiint_V dx\,dy\,dz \frac{\partial R}{\partial z}$$

How would you do it?

$$\iiint_V dx\,dy\,dz \frac{\partial R}{\partial z} = \iint_{S_{xy}} dx\,dy \int_{Z_{h_1}(x,y)}^{Z_{h_2}(x,y)} dz \frac{\partial R}{\partial z}$$

Now we write above as

$$\iint_{S_{xy}} dx\,dy\,\{R(x,y,z_{h_2}(x,y)) - R(x,y,z_{h_2}(x,y))\}$$

Now, we have $d\sigma_{xy} = dx\,dy = d\sigma\cos(\hat{n}\hat{z})$ or $d\sigma = \frac{d\sigma_{xy}}{n_z}$

Now,

$$\iint_{S_{xy}} dx\,dy\,R(x,y,z_{h_h i}(x,y))$$

is what we mean by

$$\iint_{S_{upper}} dx\,dy\,n_z\,R$$

and if we let $\hat{n}$ always mean the *outward* pointing unit vector then

$$-\iint_{S_{xy}} dx\,dy\,R(x,y,z_b(x,y)) = \iint_{S_{lower}} dx\,dy\,z_n\,R$$

So

$$\iiint_{Vol} dx\,dy\,dz\frac{\partial R}{\partial z} = \oint_S d\sigma\,n_z\,R$$

By addition

$$\iiint_{Vol} dx\,dy\,dz\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) = \oint_S d\sigma\;(\,P\,n_x\;+\;Q\,n_y\;+\;R\,n_z)$$

Writing in modern notation we have

$$\int_V dV\,\vec{\nabla}\cdot\vec{A} \;=\; \oint_S \vec{A}\cdot\hat{n}\,d\sigma$$

F. Summary of Integral Identities

At this point we collect our integral identities for easy reference.

1. The fundamental theorem of calculus:

$$\int_{x_a}^{x_b} \frac{df}{dx}dx = f(x_b) - f(x_a) \tag{8}$$

2. The fundamental theorem for line integrals:

$$\int_{\vec{r}_i}^{\vec{r}_f} d\tilde{\mathbf{r}}\cdot\overrightarrow{\nabla}F = F(\tilde{\mathbf{r}}_f) - F(\tilde{\mathbf{r}}_i) \tag{9}$$

3. Green's theorem in a plane. S_{xy} is a planar area bounded by the curve C :

$$\iint_{S_{xy}} dx\,dy\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \;=\; \oint_C (A_x\,dx + \,A_y\,dy) \tag{10}$$

4. Stokes theorem for an arbitrary vector function $\vec{A}(\vec{r})$ and a surface S bounded by a curve C:

$$\iint_S \vec{\nabla} \times \vec{A} \cdot \hat{n}\, d\sigma = \oint_C \vec{A} \cdot d\vec{r} \tag{11}$$

5. The divergence theorem for any vector function $\vec{A}(\vec{r})$, and a volume V bounded by a surface S :

$$\int_V dV\, \vec{\nabla} \cdot \vec{A} = \oint_S \vec{A} \cdot \hat{n}\, d\sigma \tag{12}$$