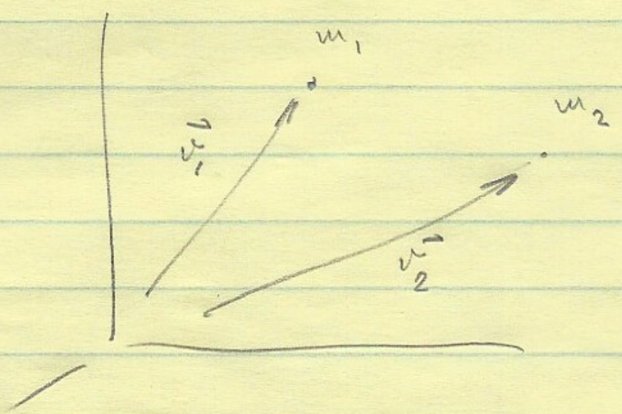
The Kepler Problem.

The Kepler orbit "gravitational problem" is one of the greatest classical problems. We address it step by step in what is a "near universal" procedure for <u>any</u> central force problem.

0.) The starting scenario.



$$\vec{r}_{12} \equiv \vec{r}_1 - \vec{r}_2$$

<u>force</u>: $$\vec{F}_{12} = -\frac{m_1 m_2 G\, \hat{r}_{12}}{|\vec{r}_{12}|^2} = -\vec{\nabla}_1 \, U_{12}(\vec{r}_{12})$$

<u>potential</u>: $$U_{12} = -\frac{m_1 m_2 G}{|\vec{r}_{12}|}$$

<u>Newton's Laws</u>:

N$_1$) $$m_1 \ddot{\vec{r}}_1 = \vec{F}_{12}$$

$$\vec{F}_{21} = -\vec{F}_{12}$$

N$_2$) $$m_2 \ddot{\vec{r}}_2 = \vec{F}_{21}$$

Our Task? ... Solve this problem!

... the steps:

1.) <u>Choose Center of mass & Relative variables.</u>

A) by <u>adding</u> N_1 & N_2 we get $m_1\ddot{\vec{r}}_1 + m_2\ddot{\vec{r}}_2 = 0$

If we define $(m_1+m_2)\vec{R}_{cm} \equiv m_1\vec{r}_1 + m_2\vec{r}_2$

then $(m_1+m_2)\ddot{\vec{R}}_{cm} = 0$ which we can integrate!

$\vec{R}_{cm} = \vec{V}_{cm}t + \vec{R}_0$ where $\vec{V}_{cm}$ & $\vec{R}_0$ are constant.

B) form: $m_2(m_1\ddot{\vec{r}}_1 = \vec{F}_{12})$ &

$m_1(m_2\ddot{\vec{r}}_2 = \vec{F}_{21})$ <u>now</u> <u>subtract</u>.

$\Longrightarrow m_1 m_2 \frac{d^2}{dt^2}(\vec{r}_1 - \vec{r}_2) = (m_1+m_2)\vec{F}_{12}$

lets simply call $\vec{r}_{12} \longleftrightarrow \vec{r}$, then

$$\left(\frac{m_1 m_2}{m_1+m_2}\right)\ddot{\vec{r}} = \vec{F}(\vec{r}) = -\frac{m_1 m_2 G}{|\vec{r}|^2}\hat{r}$$

define $\mu \equiv \left(\frac{m_1 m_2}{m_1+m_2}\right) = \left(\frac{1}{m_1}+\frac{1}{m_2}\right)^{-1}$ the "reduced mass"

We obtain:

$\mu\ddot{\vec{r}} = \vec{F}$ which "looks like" the gravitational problem for a single particle.

2.) Constants of the Motion (ie. "First Integrals")

Energy: describe our force law as $\vec{F} = -\frac{k}{r^2}\hat{r}$

where in this case $k = m_1 m_2 G$ ... but later we may use the solution for the Coulomb force too.

$\Longrightarrow \vec{F} = -\vec{\nabla}\left(-\frac{k}{r}\right)$, so using the W.E. thm. ...

$$\frac{1}{2}\mu\dot{\vec{r}}^2 + U(r) = E$$

Angular Momentum: Since $\vec{r} \times \vec{F} \equiv 0$ we have

$$\vec{r} \times \mu\ddot{\vec{r}} = 0 \text{ ... but } \vec{r} \times \ddot{\vec{r}} = \frac{d}{dt}(\vec{r} \times \dot{\vec{r}}) \text{ ... so ...}$$

$\vec{r} \times \mu\dot{\vec{r}} = \vec{L}$ a constant vector.

This implies that $\vec{r}$ & $\dot{\vec{r}}$ are both perpendicular to $\vec{L}$! ... and this is the plane $\perp$ to $\vec{L}$.

For simplicity I call this the $\hat{x}$-$\hat{y}$ plane and also call $\vec{L} = l\hat{z}$ where l is the constant angular momentum magnitude.

Thus, we could write $\vec{r} = x\hat{x} + y\hat{y}$ ... but we choose polar coordinates instead: $\vec{r} = r\hat{r}(\theta)$ where $\hat{r} = \langle\cos\theta, \sin\theta\rangle$ and $\frac{d\hat{r}}{d\theta} \equiv \hat{\theta} = \langle-\sin\theta, \cos\theta\rangle$

So! $d\vec{r} = dr\,\hat{r} + r\,d\theta\,\hat{\theta}$

3.) Cast our Energy & Angular Momentum equations in polar form.

Since $d\vec{r} = dr\,\hat{r} + r\,d\theta\,\hat{\theta}$ now divide by dt ...

$$\dot{\vec{r}} = \dot{r}\,\hat{r} + r\dot{\theta}\,\hat{\theta}$$

$$\Longrightarrow \dot{\vec{r}}^{\,2} = \dot{r}^2 + (r\dot{\theta})^2 \qquad \text{and} \ldots$$

$$\mu\,\vec{r} \times \dot{\vec{r}} = \mu r^2 \dot{\theta}\,\hat{z}$$

So! Our conservation of Energy becomes:

(a.) $$\frac{1}{2}\mu\left(\dot{r}^2 + (r\dot{\theta})^2\right) - \frac{k}{r} = E$$

Our conservation of Angular momentum is:

(b.) $$\mu r^2 \dot{\theta} = l$$

From these two equations ... everything follows.

These results are 1st order in the derivatives, but spectacularly non-linear. It is not at all obvious that any progress can be made. We will need every trick & device we own to manage the job.

4.) The Tricks.

T_1) Separate $\dot{\theta}$ out of our energy equation by using (b). Namely...

$$r\dot{\theta} = \frac{\ell}{\mu r} \quad , \ldots \text{this yields}$$

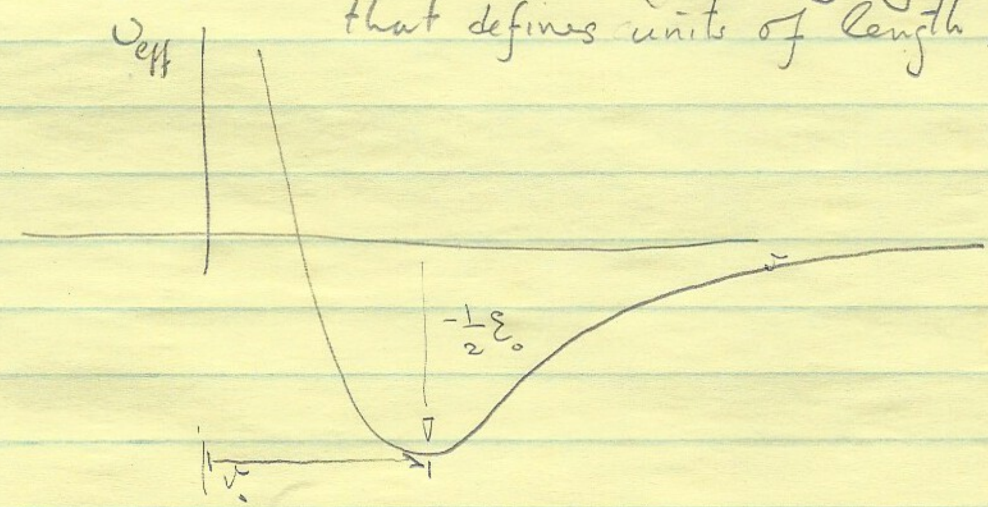
$$\frac{\mu}{2}\left(\dot{r}^2 + \left(\frac{\ell}{\mu r}\right)^2\right) - \frac{k}{r} = E$$

T_2) Now regroup terms!

$$\frac{\mu}{2}\dot{r}^2 + \left\{\frac{\ell^2}{2\mu r^2} - \frac{k}{r}\right\} = E$$

This "looks like" a single variable problem with an "effective potential" $V_{eff} = \frac{\ell^2}{2\mu r^2} - \frac{k}{r}$.

This graphs out in an easily recognizable form that defines units of length & energy.



T_3) The natural units:

$$U'_{eff}(r)\Big|^{r_0} = 0$$ defines r_0 the equilibrium distance

$$U_{eff}(r) = \left\{ \frac{\ell^2}{2\mu r^2} - \frac{k}{r} \right\}$$

$$U'_{eff}(r) = \left\{ -\frac{\ell^2}{\mu r^2} + \frac{k}{r} \right\} \frac{1}{r}$$

$$U'_{eff} = 0 \Rightarrow \left(\frac{\ell^2}{\mu r^2} = \frac{k}{r} \right)\Big|^{r_0} \quad \text{so} \quad \frac{\ell^2}{\mu k} = r_0$$

and! $\frac{\ell^2}{\mu r_0^2} = \frac{k}{r_0} \Rightarrow$ we can write a simplified statement of U_{eff}

write $r = \frac{r}{r_0} r_0 \equiv \bar{r} r_0$ so

$$U_{eff} = \left\{ \frac{\ell^2}{2\mu r_0^2 \bar{r}^2} - \frac{k}{r_0 \bar{r}} \right\} = \frac{\ell^2}{\mu_0 r_0^2} \left\{ \frac{1}{2\bar{r}^2} - \frac{1}{\bar{r}} \right\}$$

and we observe that $\frac{\ell^2}{\mu_0 r_0^2} = \frac{k}{r_0}$ defines a natural unit of Energy!

Notice that if our radius were r_0, it would be constant and the angular momentum equation would be too.

so $\mu r^2 \dot{\theta} \rightarrow \mu r_0^2 \dot{\theta}_0 = \ell$ and $\mu r_0^2 \dot{\theta}_0^2 = \frac{\ell^2}{\mu r_0^2}$

So $\frac{\ell^2}{\mu r_0^2} = \frac{k}{r_0} = \mu r_0^2 \dot{\theta}_0^2 \equiv \mathcal{E}_0$ our energy unit!

Our energy equation now appears as :

$$\frac{\mu}{2}\dot{r}^2 = \mathcal{E}_0\left(\frac{1}{2}\frac{1}{\bar{r}^2} - \frac{1}{\bar{r}}\right) = E$$

T_4) Let "θ" be our "independent" variable!

How? Write $\dot{r} = \frac{dr}{dt} = \frac{d\theta}{dt}\frac{dr}{d\theta} \equiv \dot{\theta}\, r'$

but $\dot{\theta} = \frac{\ell}{\mu r^2}$ so $\dot{r} = \frac{\ell}{\mu}\frac{1}{r^2}r' = -\frac{\ell}{\mu}\frac{d}{d\theta}\frac{1}{r}$

$$\Longrightarrow \frac{1}{2}\mu\dot{r}^2 = \frac{1}{2}\mu\frac{\ell^2}{\mu^2}\left(\frac{d}{d\theta}\frac{1}{r}\right)^2 = \frac{1}{2}\frac{\ell^2}{\mu r_0^2}\left[\frac{d}{d\theta}\left(\frac{r_0}{r}\right)\right]^2$$

or $\frac{1}{2}\mu\dot{r}^2 = \frac{1}{2}\mathcal{E}_0\left(\frac{d}{d\theta}\frac{1}{\bar{r}}\right)^2$

So at last we have $\frac{1}{2}\mathcal{E}_0\left(\frac{d}{d\theta}\frac{1}{\bar{r}}\right)^2 + \mathcal{E}_0\left\{\frac{1}{2}\frac{1}{\bar{r}^2} - \frac{1}{\bar{r}}\right\} = E$

Which gives $\left(\frac{d}{d\theta}\frac{1}{\bar{r}}\right)^2 + \left\{\frac{1}{\bar{r}^2} - \frac{2}{\bar{r}}\right\} = 2\frac{E}{\mathcal{E}_0}$

T_5) Now add "1" to both sides! since $\frac{1}{\bar{r}^2} - \frac{2}{\bar{r}} + 1 = \left(\frac{1}{\bar{r}} - 1\right)^2$

and $\frac{d}{d\theta}\frac{1}{\bar{r}} = \frac{d}{d\theta}\left(\frac{1}{\bar{r}} - 1\right)$, yielding

$$\left[\frac{d}{d\theta}\left(\frac{1}{\bar{r}} - 1\right)\right]^2 + \left(\frac{1}{\bar{r}} - 1\right)^2 = 1 + 2\frac{E}{\mathcal{E}_0}$$ ... now call $\frac{1}{\bar{r}} - 1 = u$

$\Longrightarrow$ $u'^2 + u^2 = 1 + 2\frac{E}{\mathcal{E}_0} \equiv \epsilon^2$ which has a simple solution!

Apparently $u = \epsilon\cos(\theta - \theta_0)$

we conclude: $\frac{1}{r} = 1 + \epsilon\cos(\theta - \theta_0)$

This is our ellipse! (or hyperbola!)

5.) The Time Dependence.

So now we have $r = f(\theta)$ — next we need $\theta = g(t)$. This comes from our angular momentum equation. $\mu r^2 \dot{\theta} = l$

For circular motion this would be

$$\mu r_0^2 \dot{\theta}_0 = l \quad \text{and taking ratios} \ldots$$

$$\dot{\theta} = \dot{\theta}_0 \frac{1}{r^2} = \dot{\theta}_0 \left(1 + \epsilon\cos(\theta)\right)^2$$

so $$\frac{d\theta}{(1+\epsilon\cos\theta)^2} = \dot{\theta}_0\, dt$$

In principle, we merely integrate and are done! Unfortunately, this is a tough integral!!

The prefered method is to use the Kepler construction contained in the companion notes. There we concluded that:

$$\frac{t}{T_{period}} = \frac{1}{2\pi}\left[\psi - \epsilon\sin\psi\right] \quad \text{where}$$

$$\tan\left(\frac{\theta}{2}\right) = \sqrt{\frac{1+\epsilon}{1-\epsilon}}\,\tan\left(\frac{\psi}{2}\right)$$ is a parametric relation between $t \leftrightarrow \theta$.