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I. *NOTES ON ORTHOGONAL CURVILINEAR COORDINATES*

A. Introduction

Locating a unique position in 3-D space requires choosing three numbers or "coordinates" for its specification. Although using Cartesian coordinates is, by far, the most common and familiar choice, coordinates may actually be chosen in a very wide variety of ways. In general then, let us signify our choice simply as the set of three numbers $\{q_1, q_2, q_3\}$. We understand implicitly, of course, that any selection of coordinates can be transformed into any other set by direct algebraic transformation. That is, all choices are equivalent in content.

Next, we observe that if we hold any two of the 'q-coordinates' constant, but now increase the third slightly (i.e. infinitesimally) ... that the position vector starts to move "infinitesimally" along a 'q-curve.' The respective tangents to the 3 possible 'q-curves' which emerge from any given point now define three distinct directions. In the general case, these directions need not be perpendicular to each other and that introduces the study of generalized curvilinear coordinates. However, in those special cases that these three directions **are** always mutually orthogonal, we say that we are dealing with an 'orthogonal curvilinear coordinate system.' Such systems really are quite common and have such a pronounced utility that they are well worth our detailed study.

Now notice that if we make simultaneous infinitesimal increases in the coordinates (i.e. $q_1 \to q_1 + dq_1, q_2 \to q_2 + dq_2, q_3 \to q_3 + dq_3$) that the position vector suffers an infinitesimal displacement $\vec{r} \to \vec{r} + d\vec{r}$ where

$$d\vec{r} = \frac{\partial \vec{r}}{\partial q_1}\, dq_1 + \frac{\partial \vec{r}}{\partial q_2}\, dq_2 + \frac{\partial \vec{r}}{\partial q_3}\, dq_3$$

The vectors $\frac{\partial \vec{r}}{\partial q_i}$ are along the tangent direction mentioned above — but they are not of unit length. We introduce new symbols to make their lengths explicit. We define the symbol 'h_i' to be the length of $\frac{\partial \vec{r}}{\partial q_i}$ thus $h_i = \sqrt{\frac{\partial \vec{r}}{\partial q_i} \cdot \frac{\partial \vec{r}}{\partial q_i}}$ and we may finally write $\frac{\partial \vec{r}}{\partial q_i} = h_i \hat{a}_i$ where $\hat{a}_i$ is the unit tangent vector to the i^{th} q—curve. *Now summarize*: We have

$$d\vec{r} = \frac{\partial \vec{r}}{\partial q_1}\, dq_1 + \frac{\partial \vec{r}}{\partial q_2}\, dq_2 + \frac{\partial \vec{r}}{\partial q_3}\, dq_3$$

which may be written

$$d\vec{r} = \hat{a}_1\, h_1\, dq_1 + \hat{a}_2\, h_2\, dq_2 + \hat{a}_3\, h_3\, dq_3$$

where

$$h_i^2 = \frac{\partial \vec{r}}{\partial q_1} \cdot \frac{\partial \vec{r}}{\partial q_1}$$

Since we are discussing orthogonal coordinates we have $\hat{a}_i \cdot \hat{a}_j = \delta_{ij}$ and $\hat{a}_i \times \hat{a}_j = \sum_k \epsilon_{ijk} \hat{a}_k$.

1. *Discussion*

First note that $h_i\, dq_i$ always has the dimension of length — indeed it is the physical distance that the $\vec{r}$—vector moves through as we change $q_i \to q_i + dq_i$ holding the other two coordinates constant.

Ultimately, all operations (e.g. gradient, divergence, curl ...) can be known entirely in terms of the 3 h_i — thus they play a central role for curvilinear orthogonal coordinates.

B. Vector Differential Operations in Curvilinear Coordinates

1. Basic Definitions

Start from basic definitions: $\frac{\partial \vec{r}}{\partial q_i} = h_i\, \hat{a}_i$ and dot both sides with $\hat{e}_j$ and obtain $\frac{\partial x_j}{\partial q_i} = h_i\, \hat{e}_j \cdot \hat{a}_i$. We may write $\hat{e}_k = \sum_i \hat{a}_i \left(\frac{1}{h_i} \frac{\partial x_i}{\partial q_i}\right)$

2. Gradient

We know the gradient in Cartesian coordinates $\vec{\nabla} = \sum_k \hat{e}_k \frac{\partial}{\partial x_k}$. Now we simply transform to 'q-coordinates' using $\frac{\partial}{\partial x_i} = \sum_j \frac{\partial q_j}{\partial x_k} \frac{\partial}{\partial q_j}$.

Now combine with the above to obtain

$$\begin{aligned} \nabla &= \sum_k \left(\sum_i \hat{a}_i \frac{1}{h_k} \frac{\partial x_k}{\partial q_i}\right) \left(\sum_j \frac{\partial q_j}{\partial x_k} \frac{\partial}{\partial q_j}\right) \\ &= \sum_{ij} \hat{a}_i \frac{1}{h_i} \left(\sum_k \frac{\partial x_k}{\partial q_i} \frac{\partial q_j}{\partial x_k}\right) \frac{\partial}{\partial q_j} \end{aligned}$$

but $\sum_k \frac{\partial x_k}{\partial q_i} \frac{\partial q_j}{\partial x_i} = \frac{\partial q_j}{\partial q_i} = \delta_{ij}$ So we recover

$$\nabla = \sum_i \hat{a}_i \frac{1}{h_i} \frac{\partial}{\partial q_i}$$

3. Curl

Start with the observation that: $\hat{a}_i = h_i\, \nabla q_i$ (by inspection) so that $\nabla \times \hat{a} = \nabla\, h_i \times \nabla q_i + h_i\, \nabla \times \nabla q_i$ which implies that $\nabla \times \hat{a}_i = \nabla h_i \times \nabla q_i = \frac{1}{h_i} \nabla h_i \times \hat{a}_i$ Since any vector $\vec{A}$ may be expressed as $\vec{A} = \sum_i A_i\, \hat{a}_i$ Hence we have

$$\nabla \times \vec{A} = \sum_i (\nabla A_i + \hat{a}_i + A_i \nabla \times \hat{a}_i)$$

or

$$\nabla \times \vec{A} = \sum_i \frac{1}{h_i}\, \nabla(h_i\, A_i) \times \hat{a}_i$$

but also

$$\nabla = \sum_k \hat{a}_k \frac{1}{h_k} \frac{\partial}{\partial q_k}$$

so we write

$$\nabla \times \vec{A} = \sum_{ik} \frac{1}{h_i} \frac{1}{h_k} \frac{\partial}{\partial q_i} (h_i A_i) \hat{a}_k \times \hat{a}_i$$

but

$$\hat{a}_k \times \hat{a}_i = \sum_j \epsilon_{ijk}\, \hat{a}_k$$

so

$$\nabla \times \vec{A} = \sum_{ijk} \epsilon_{ijk} \frac{1}{h_i h_k} \frac{\partial}{\partial q_k} (h_i A_i) \hat{a}_j$$

. Finally, we have then

$$\nabla \times \vec{A} = \sum_{ijk} \epsilon_{kij} \frac{1}{h_i h_j h_k}\, h_j\, \hat{a}_j \frac{\partial}{\partial q_k}(h_i A_i).$$

Which is commonly written

$$\nabla \times \vec{A} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{a}_1 & h_2 \hat{a}_2 & h_3 \hat{a}_3 \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}$$

4. Divergence

Use the general vector identity

$$\nabla \cdot (A \times B) = B \cdot \nabla \times A - A \cdot \nabla \times B$$

where we apply it to the identity

$$\hat{a}_3 = \hat{a}_1 \times \hat{a}_2.$$

So

$$\begin{aligned} \nabla \cdot \hat{a}_3 &= \hat{a}_2 \cdot \nabla \times \hat{a}_1 - \hat{a}_1 \cdot \nabla \times \hat{a}_2 \\ &= \hat{a}_2 \cdot \nabla \times \hat{a}_1 - \hat{a}_1 \cdot \left(\frac{1}{h_2} \nabla h_2 \times \hat{a}_2 \right) \\ &= \frac{1}{h_1 h_3} \frac{\partial h_1}{\partial q_3} + \frac{1}{h_2 h_3} \frac{\partial h_2}{\partial q_3} \\ &= \frac{1}{h_1 h_2 h_3} \frac{\partial (h_1 h_2)}{\partial q_3} \end{aligned}$$

Then since $\vec{A} = \sum_i A_i\, \hat{a}_i$ we can write

$$\begin{aligned} \nabla \cdot \vec{A} &= \sum_i \left(\nabla A_i \cdot \hat{a}_i + A_i\, \nabla \cdot \hat{a}_i \right) \\ &= \sum_i \left(\frac{1}{h_i} \frac{\partial A_i}{\partial q_i} + A_i \nabla \cdot \hat{a}_i \right) \\ &= \frac{1}{h_1 h_2 h_3} \left(\frac{\partial}{\partial q_1}(A_1\, h_2\, h_3) + \frac{\partial}{\partial q_2}(A_2\, h_3\, h_1) + \frac{\partial}{\partial q_3}(A_3\, h_1\, h_3) \right) \end{aligned}$$

Finally, if $\vec{A} = \nabla \phi$ we have

$$\nabla^2 \phi = \frac{1}{h_1\, h_2\, h_3} \left(\frac{\partial}{\partial q_1} \left(\frac{h_2\, h_3}{h_1} \frac{\partial \phi}{\partial q_1} \right) + \frac{\partial}{\partial q_2} \left(\frac{h_3 h_1}{h_2} \frac{\partial \phi}{\partial q_2} \right) + \frac{\partial}{\partial q_3} \left(\frac{h_1 h_2}{h_3} \frac{\partial \phi}{\partial q_3} \right) \right)$$

C. Curvilinear Coordinates II

1. A Second Approach to Differential Operators

We could deduce all our identities in a much more straight forward and simple way if only we knew the generic derivative $\frac{\partial \hat{a}_i}{\partial q_j}$. Remarkably, this is fairly difficult to find!

Insight into how to do it can be gained from watching basic geometric properties. Since $\{\hat{a}_1, \hat{a}_2, \hat{a}_3\}$ are always perpendicular and always of unit length, the manner in which they change together is highly restricted. In particular, we must *always* preserve:

$$\hat{a}_i \cdot \hat{a}_j = \delta_{ij}.$$

Notice that in any infinitesimal change, then, we must have: $d\hat{a}_i \cdot \hat{a}_j + \hat{a}_i \cdot d\hat{a}_j = 0$, and so $d\hat{a}_i \cdot \hat{a}_i = 0$ This condition preserves all our unit unit lengths.
Altogether, these imply, then:

$$\begin{aligned} d\hat{a}_1 &= \hat{a}_2 \, (\hat{a}_2 \cdot d\hat{a}_1) + \hat{a}_3 \, (\hat{a}_3 \cdot d\hat{a}_1) \\ d\hat{a}_1 &= \hat{a}_3 \, (\hat{a}_3 \cdot d\hat{a}_2) + \hat{a}_1 \, (\hat{a}_1 \cdot d\hat{a}_2) \\ d\hat{a}_3 &= \hat{a}_1 \, (\hat{a}_1 \cdot d\hat{a}_3) + \hat{a}_2 \, (\hat{a}_2 \cdot d\hat{a}_3) \end{aligned}$$

Since *any* vector (even infinitesimal ones) can be expanded on the three basic vectors. Now let's simply define the numbers:

$$\{d\theta_1, d\theta_2, d\theta_3\}$$

by:

$$\begin{aligned} d\theta_1 &= \hat{a}_3 \cdot d\hat{a}_2 \; (= -d\hat{a}_3 \cdot \hat{a}_2) \\ d\theta_2 &= \hat{a}_1 \cdot d\hat{a}_3 \; (= -d\hat{a}_1 \cdot \hat{a}_3) \\ d\theta_3 &= \hat{a}_2 \cdot d\hat{a}_1 \; (= -d\hat{a}_2 \cdot \hat{a}_1) \end{aligned}$$

and then we can write

$$\begin{aligned} d\hat{a}_1 &= d\theta_3 \, \hat{a}_2 - d\theta_2 \, \hat{a}_3 \\ d\hat{a}_2 &= d\theta_1 \, \hat{a}_3 - d\theta_3 \, \hat{a}_1 \\ d\hat{a}_3 &= d\theta_2 \, \hat{a}_1 - d\theta_1 \, \hat{a}_2 \end{aligned}$$

which can be summarized by using the vector $d\vec{\theta}$ defined by

$$d\vec{\theta} = d\theta_1 \hat{a}_1 + d\theta_2 \, \hat{a}_2 + d\theta_3 \, \hat{a}_3$$

so that

$$\begin{aligned} d\hat{a}_1 &= d\vec{\theta} \times \hat{a}_1 \\ d\hat{a}_2 &= d\vec{\theta} \times \hat{a}_2 \\ d\hat{a}_3 &= d\vec{\theta} \times \hat{a}_3 \end{aligned}$$

which is simple and physical. This result has the compelling interpretation that any change in orientation of our orthogonal triple $(\hat{a}_1, \hat{a}_2, \hat{a}_3)$ is, in effect, a *rotation*. The vector $d\vec{\theta}$ is our rotation vector. We write, then, in general:

$$d\hat{a}_i = d\vec{\theta} \times \hat{a}_i$$

We cross the definition with $\hat{a}_i$ and sum over i to achieve

$$\begin{aligned} \sum_i \hat{a}_\times d\hat{a}_i &= \sum_i \hat{a}_i \times (d\vec{\theta} \times \hat{a}_i) \\ &= \sum_i (d\vec{\theta} - \hat{a}_i \, \hat{a}_i \cdot d\vec{\theta}) \\ &= 2d\vec{\theta} \end{aligned}$$

then

$$d\vec{\theta} = \frac{1}{2}\sum_i \hat{a}_i \times d\hat{a}_i$$

We can now find the right hand side from considering the following basic definition

$$\frac{\partial \vec{r}}{\partial q_i} = h_i\,\hat{a}_i.$$

So first

$$\begin{aligned}
\vec{\nabla} \times \frac{\partial \vec{r}}{\partial q_i} &= \vec{\nabla} \times (h_i\,\hat{a}_i) \\
&= \vec{\nabla} h_i \times \hat{a}_i + h_i\,\vec{\nabla} \times \hat{a}_i \\
&= 2\,\vec{\nabla} h_i \times \hat{a}_i
\end{aligned}$$

But also ...

$$\begin{aligned}
\vec{\nabla} \times \frac{\partial \vec{r}}{\partial q_i} &= \sum_j \left(\hat{a}_j \times \frac{1}{h_j}\frac{\partial}{\partial q_j}\,\frac{\partial \vec{r}}{\partial q_i}\right) \\
&= \sum_j \left(\hat{a}_j \times \frac{1}{h_j}\,\frac{\partial}{\partial q_i}\,\frac{\partial \vec{r}}{\partial q_j}\right) \\
&= \sum_j \hat{a}_j \times \frac{1}{h_j}\left(\frac{\partial h_j}{\partial q_i}\,\hat{a}_j + h_j\,\frac{\partial \hat{a}_j}{\partial q_i}\right) \\
&= \sum_j \hat{a}_j \times \frac{\partial \hat{a}_j}{\partial q_i}
\end{aligned}$$

So

$$\sum_j \hat{a}_j \times \frac{\partial \hat{a}_j}{\partial q_i} = 2\,\vec{\nabla}\,h_i \times \hat{a}_i$$

and now ... Since

$$\sum_i dq_i\,\frac{\partial \hat{a}_j}{\partial q_i} = d\hat{a}_j$$

we multiply above by dq_i and sum over i to conclude

$$\frac{1}{2}\sum_j \hat{a}_j \times d\hat{a}_j = \sum_i dq_i\,\vec{\nabla} h_i \times \hat{a}_i = d\vec{\theta}$$

D. Summary

We conclude that an orthogonal triad of unit vectors must change in a very restricted way - in fact it must change as a *rotation*. The key idea has been to identify the rotation vector $d\vec{\theta}$ defining that rotation. Once we have done that every thing else follows as we summarize next. We have shown that in *any* infinitesimal change there exists $d\vec{\theta}$ such that:

$$\begin{aligned} d\hat{a}_i &= d\vec{\theta} \times \hat{a}_i \qquad \textit{where} \\ d\vec{\theta} &= \sum_j dq_j \, \vec{\nabla} h_j \times \hat{a}_j \end{aligned}$$

So, for example, if we specify that only one particular coordinate, say q_j, changes i.e. $q_j \rightarrow q_j + dq_j$, then

$$d\hat{a}_i = dq_j \; (\vec{\nabla} h_j \times \hat{a}_j) \times \hat{a}_i$$

or, as we finally conclude,

$$\frac{\partial \hat{a}_i}{\partial q_j} = (\vec{\nabla} h_j \times \hat{a}_j) \times \hat{a}_i$$

and from this result we may build all of the other vector differential identities in a straight forward manner.