CSUC
Department of Physics
301A/B Mechanics

I. *NOTES ON ROTATIONS*

A. Rotations and the Rotation Matrix

The study of rotations may be approached from either the "active" or "passive" points of view. An "active" rotation is one where physical vectors are rotated about some axis and we follow their changing components. A "passive" rotation is one where the physical vectors remain at rest but the coordinate axes are rotated. Clearly, the changing relationships of vectors to coordinate axes can be represented either way.

Any rotation may be characterized by a rotation axis $\hat{n}$ and a rotation angle θ about this axis. Consider an arbitrary rotation of our coordinate system (i.e. suppose for now we adopt the "passive" point of view). The original set of coordinate axes $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ are rotated into a new set

$$(\hat{e}_1^{'}, \hat{e}_2^{'}, \hat{e}_3^{'}).$$

For a given vector $\vec{A}$ how do the new and old components relate?

$$A_i = \hat{e}_i \cdot \vec{A} \; and \; A_i^{'} = \hat{e}_i^{'} \cdot \vec{A}$$

I relate these via

$$A_i^{'} = \hat{e}_i^{'} \cdot \left(\sum_j \hat{e}_j \, \hat{e}_j \cdot \vec{A} \right) = \sum_j \hat{e}_i^{'} \cdot \hat{e}_j \, A_j$$

This relationship is often displayed in an equivalent matrix form

$$\begin{pmatrix} A_1^{'} \\ A_2^{'} \\ A_3^{'} \end{pmatrix} = \begin{pmatrix} \hat{e}_1^{'} \cdot \hat{e}_1 & \hat{e}_1^{'} \cdot \hat{e}_2 & \hat{e}_1^{'} \cdot \hat{e}_3 \\ \hat{e}_2^{'} \cdot \hat{e}_1 & \hat{e}_2^{'} \cdot \hat{e}_2 & \hat{e}_2^{'} \cdot \hat{e}_3 \\ \hat{e}_3^{'} \cdot \hat{e}_1 & \hat{e}_3^{'} \cdot \hat{e}_2 & \hat{e}_3^{'} \cdot \hat{e}_3 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}$$

This defines the *rotation matrix* R_{ij} where, then,

$$A_i^{'} = \sum_j R_{ij} A_j$$

$R_{ij} = \hat{e}_i^{'} \cdot \hat{e}_j$ carries all our geometric preconceptions about just what rotations in 3-D space are like. Correspondingly, in studying rotations we can do it all through examination of its properties. First, rotations preserve dot products

$$\sum_i A_i^{'} B_i^{'} = \sum_i A_i \, B_i$$

Proof: Let

$$A_i^{'} = \sum_j R_{ij} A_j \; so \; \sum_i A_i^{'} B_i^{'} = \sum_{jk} R_{ij} R_{ik} A_j B_k$$

But

$$\sum_i R_{ij}R_{ik} = \sum_i \hat{e}_i^{'}\hat{e}_j\,\hat{e}_i^{'}\hat{e}_k = \sum_i \hat{e}_j \cdot \hat{e}_i^{'}\,\hat{e}_i^{'} \cdot \hat{e}_k = \hat{e}_j \cdot \hat{e}_k = \delta_{jk}$$

Which implies that we have

$$\sum_{jk} \delta_{jk} A_j B_k = \sum_j A_j B_j$$

In summary, then:

$$\sum_i R_{ij}R_{ik} = \sum_i R_{ji}R_{ki} = \delta_{jk}$$

Second, the determinant $det\ R_{ij} = 1$ which we comes from the requirement

$$\hat{e}_1^{'} \cdot (\hat{e}_2^{'} \times \hat{e}_3^{'}) = 1$$

and exploits the fact that

$$\hat{e}_i \cdot (\hat{e}_j \times \hat{e}_k) = \epsilon_{ijk}$$

I use

$$\hat{e}_i^{'} = \sum_i \hat{e}_i \hat{e}_i \cdot \hat{e}_i^{'} = \sum R_{ij}\hat{e}_i$$

So

$$\hat{e}_1^{'} \cdot (\hat{e}_2^{'} \times \hat{e}_3^{'}) = \sum_{ijk} R_{1i}R_{2j}R_{3k}\hat{e}_i \cdot (\hat{e}_j \times \hat{e}_k) = \sum_{ijk} R_{1i}R_{2j}R_{3k}\epsilon_{ijk} = det(R)$$

So $1 = det(R_{ij})$.

Third, the rotation matrices have the multiplication property; i.e. the matrix product of two rotation matrices yields the correct matrix for the combined rotation process. Suppose

$$\{\hat{e}_i\} \rightarrow \{\hat{e}_i^{'}\} \rightarrow \hat{e}_i^{''}$$

where each transformation is referenced by

$$R_{1_{ij}} = \hat{e}_i^{'} \cdot \hat{e}_j \ and \ R_{2_{ij}} = \hat{e}_i^{''} \cdot \hat{e}_j^{'}$$

and

$$(R_2R_1)_{ij} = \sum_k R_{2_{ik}} R_{1_{kj}} = \sum_k \hat{e}_i^{''} \cdot \hat{e}_k^{'}\,\hat{e}_k^{'} \cdot \hat{e}_j = \hat{e}_i^{''} \cdot \hat{e}_j$$

which is indeed the correct element for transforming from

$$\{\hat{e}_i\} \rightarrow \{\hat{e}_i^{''}\}$$

directly.

B. Alternative Representations of The Rotation Matrix

We have seen that the rotation matrix may be represented via dot products of new and old coordinate axes, i.e.

$$R_{ij} = \hat{e}_i^{'} \cdot \hat{e}_j$$

A further representation originates in the notion of building up a finite rotation via a succession of infinitesimal rotations. Suppose then, that we rotate a vector $\vec{v}$ about an axis $\hat{n}$ by an angle $\delta\theta$. The change in the vector is

$$\delta\vec{v} = \delta\theta\hat{n} \times \vec{v}$$

We write

$$\vec{v}_{new} = \vec{v} + \delta\vec{v} = \vec{v} + \delta\theta\hat{n} \times \vec{v}$$

which we may write as

$$\vec{v}_{new} = (1 + \delta\theta\hat{n}\times)\vec{v}$$

Suppose now that a finite rotation of θ about $\hat{n}$ may be achieved as a large number "N" of rotations each of magnitude $\frac{\theta}{N}$. We write,

$$\vec{v}_{new} = \lim_{N\to\infty}(1 + \frac{\theta}{N}\hat{n}\times)^N\,\vec{v} = (\exp(\theta\,\hat{n}\,\times))\,\vec{v}$$

To see that this is well defined, consider expanding the exponential in its power series

$$\exp(\theta\hat{n}\times) = 1 + \frac{\theta^1}{1!}(\hat{n}\times)^1 + \frac{\theta^2}{2!}(\hat{n}\times)^2 + \cdots$$

This is simpler than it looks if we observe that

$$(\hat{n}\times)^3 = -\hat{n}\times$$

So the infinite series may be rearranged as 1 + (odd powers) + (even powers)

$$\begin{gathered} 1 + \left(\tfrac{\theta^1}{1!}(\hat{n}\times)^1 + \tfrac{\theta^3}{3!}(\hat{n}\times)^3 + \cdots\right) + \left(\tfrac{\theta^2}{2!}(\hat{n}\times)^2 + \tfrac{\theta^4}{4!}(\hat{n}\times)^4 + \cdots\right) = \\ 1 + \left(\theta - \tfrac{\theta^3}{3!} + \cdots\right)\hat{n}\,\times + \left(\tfrac{\theta^2}{2!} - \tfrac{\theta^4}{4!} + \cdots\right)(\hat{n}\times)^2 = \\ 1 + \sin(\theta)\hat{n}\,\times + (1 - \cos(\theta))(\hat{n}\times)^2 \end{gathered}$$

This is the vector representation of the rotation matrix. It may easily be translated into ordinary matrix form if we recall that

$$\hat{n} \times \vec{A} = \begin{pmatrix} 0 & -\hat{n}_z & \hat{n}_y \\ \hat{n}_z & 0 & -\hat{n}_x \\ -\hat{n}_y & \hat{n}_x & 0 \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

We actually derived this form in the class by breaking a vector into components parallel and perpendicular to the rotation axis $\hat{n}$.

In that case

$$\vec{v}_{new} = \vec{v}_{\parallel} + \vec{v}_{\perp} \cos(\theta) + \hat{n} \times \vec{v}_{\perp} \sin(\theta)$$

To make the connection just use

$$\hat{n} \times (\hat{n} \times \;) = (\hat{n}\hat{n} - 1)$$

and then

$$\begin{aligned} & 1 + \sin(\theta)(\hat{n} \times \;) + (1 - \cos(\theta))(\hat{n} \times \;)^2 \\ = \; & 1 + \sin(\theta)(\hat{n} \times \;) + (1 - \cos(\theta))(\hat{n}\hat{n} \cdot -1) \\ = \; & \hat{n}\hat{n} + \cos(\theta)(1 - \hat{n}\hat{n} \cdot) + \sin(\theta)\,(\hat{n} \times \;) \end{aligned}$$

which applied to any vector $\vec{v}$ does, indeed, give

$$\vec{v}_{\parallel} + \cos(\theta)\,\vec{v}_{\perp} + \sin(\theta)\,\hat{n} \times \vec{v}_{\perp}$$