CSUC
Department of Physics
301A Mechanics:

Notes on Dimensional Analysis and Units

I. INTRODUCTION

Physics has a particular interest in those attributes of nature which allow *comparison processes*. Such qualities include e.g. length, time, mass, force, energy...., etc. By "comparison process" we mean that although we recognize that, say, "length" is an *abstract* quality, nonetheless, any two "lengths" may successfully be **compared** in a generally agreed upon manner yielding a *real number* which we sometimes, by custom, call their "ratio". It's crucial to realize that this outcome it isn't really a mathematical *ratio* since the participants aren't numbers at all, but rather completely abstract entities ... viz. "*Physical Extent in Space*". Thus we undertake to represent physical qualities (i.e. certain aspects) of our system with real numbers. In general, we recognize that only 'lengths' can be compared to 'other lengths' etc - i.e. that the world divides into exclusive classes (e.g. 'all lengths') of entities which can be compared to each other. We say that all the entities in any one class are 'mutually comparable' with each other. These physical numbers will be our basic tools for expressing the relationships we observe in nature. As we proceed in our understanding of the physical world we will come to understand that any (and ultimately *every !*) properly expressed relationship (e.g. our *theories*) has several universal aspects of structure which we will study using what we may call "*dimensional analysis*". These structural aspects are of enormous importance and will be, as you progress, among the very first things you examine in any physical problem you encounter. Of course, as you have already learned in your foundational classes, we, too, ***will*** find ourselves introducing the "standard-comparison-amounts" or "unit amounts" with which you are familiar. It will be a central task and aim of these notes, however, to show you that "unit amounts" play only an intermediate and sociological function (principally human communication) ... but really have *no other role* to play in physics. We might say that they have no '*physical* content' whatever. This is a great shock to those who haven't proceeded very far in their physical studies! In fact, as we will soon find out, all "*units*" will utterly disappear from any well expressed relationship and be replaced by comparisons between quantities having their origin solely inside the problem itself. We will express this by saying that **every** physical problem determines its own intrinsic <u>*natural units*</u>. We might begin (merely by casual reflection) by noting here that the choice of units is *completely arbitrary* - while **nothing** important in "Nature" is arbitrary at all (and must therefore disappear from any well crafted expression ...) !

II. LINEAR COMPARISON PROCESSES / UNITS ... AND *CHANGING* UNITS

These notes will attempt consistency of notation, and I have chosen *square brackets* e.g. $[X]$ to designate 'abstract entities' out there in *Nature* (i.e. these are **not** numbers ...). These are the abstract *things* we find out there in the world and want to talk about ... such as a *"physical extent of space"*. Any 'comparison process' results, then, in a 'real number' which is subject to the usual laws of algebra and will be represented by symbols *without* brackets, e.g. w.

Let the collection of symbols

$$\left[\frac{X}{Y}\right] = w$$

denote that the abstract length of $[X]$ has (somehow) been compared with the abstract length $[Y]$ yielding the real number w. We say w is the *measure* of $[X]$ with respect to $[Y]$. The symbol $\left[\frac{X}{Y}\right]$ indeed *looks* like a "ratio" but <u>isn't</u> because the things inside aren't numbers! The comparison process does, however, suggest the interior notion of just *how many times* the abstract quantity $[Y]$ *fits into* the abstract quantity $[X]$ and that is the numerical ratio idea too. This is the essence of *Theoretical Physics (!)* which attempts to *map* (or "overlay") mathematical structures onto our perceptions of Nature. Notice, especially, that no notion of *unit* is ever needed here! Please be aware that there is no difference in kind what-so-ever in comparing our length to another length ... and the act of comparing our length to some arbitrarily chosen unit length. Unit lengths are not special in any way ... they are just another length. The problem will turn out to be that, if we can choose <u>one</u> arbitrary unit ... then we can, at a later time (and on a whim...), choose yet <u>another</u> arbitrary one ... and we must then relate those two outcomes. Let's begin the discussion.

A. The Properties of Linear Comparison Processes

Suppose, then, we let the collection of symbols:

$$\left[\frac{X}{U_L}\right] = x$$

denote that the abstract length $[X]$ has been compared with the abstract-standard-comparison-amount $[U_L]$ yielding the real number x. We say x is the *measure* of $[X]$ with respect to $[U_L]$. If we had used a different standard comparison amount (unit), say $[U'_L]$, we would have obtained a <u>different</u> real number x' for the <u>same</u> $[X]$. Clearly, *different numbers* may represent the **same** chosen aspect of the very *same* physical entity $[X]$. This is where the confusion enters. We need to relate x and x'.

For reasons of simplicity most (*not all*!) of the comparison processes in general use today allow us to relate the numbers obtained using different units in the following highly convenient stylized manner - and we may call all such processes *Linear Measures.* If $[U_L]$ and $[U'_L]$ are two such unit amounts, then a **Linear Measuring Process** is one such that the following is taken to be true:

$$x' = \left[\frac{X}{U'_L}\right] = \left[\frac{X}{U_L}\right]\left[\frac{U_L}{U'_L}\right] = x\left[\frac{U_L}{U'_L}\right] \tag{1}$$

So that $x' = x\ \left[\frac{U_L}{U'_L}\right]$. We generally call $\left[\frac{U_L}{U'_L}\right]$ a "conversion factor." Our <u>**foundational starting rule**</u> may be summarized as follows :

Rule 0)

$$(new\#) = (old\#)\left[\frac{Old\,Unit}{New\,Unit}\right] \tag{2}$$

A concrete example of this would be: $32 = 9.8\left[\frac{meter}{foot}\right] = 9.8 \cdot (3.28)$.

These properties are so much in accord with customary practice that they rarely merit mention. Yet, one should bear in mind that they are true *only* because we carefully chose a specific *kind* of comparison process. Actually, we needn't have used special "unit amounts" in our definition of Linear Measures. We only did that to make the discussion familiar. So let us take a tiny step back and enumerate now in full generality the simple properties of **Linear Measuring Processes**.

To get the conversation going, we let $[X]$, $[Y]$, $[Z]$ denote any mutually comparable quantities. We then have:

Rule 1)

$$\left[\frac{X}{Y}\right] = \left[\frac{X}{Z}\right]\left[\frac{Z}{Y}\right] \qquad for\ any \quad [Z] \tag{3}$$

Rule 2)

$$1 = \left[\frac{X}{X}\right] \qquad \ldots and\ so\ with\ Rule\,1)\ \ldots \qquad 1 = \left[\frac{X}{Y}\right]\left[\frac{Y}{X}\right] \qquad implying \qquad \left[\frac{Y}{X}\right] = \left(\left[\frac{X}{Y}\right]\right)^{-1} \tag{4}$$

And now using <u>both</u> rules above, it follows that we have:

Rule 3)

$$\left[\frac{X}{Y}\right] = \left[\frac{X}{Z}\right]\left[\frac{Z}{Y}\right] = \frac{\left[\frac{X}{Z}\right]}{\left[\frac{Y}{Z}\right]} \qquad for\ any \quad [Z] \tag{5}$$

And now using Rule 1) twice we observe:

Rule 4)

$$\left[\frac{X}{Y}\right] = \left[\frac{X}{Z_1}\right]\left[\frac{Z_1}{Y}\right] = \left[\frac{X}{Z_2}\right]\left[\frac{Z_2}{Y}\right] \qquad for\ any \quad [Z_1]\ and\ [Z_2] \tag{6}$$

Now this is very useful because if we were to choose for $[Z_1]$ and $[Z_2]$ two unit amounts, say: $[U_1]$ and $[U_2]$, and if we now denote $x_1 \equiv \left[\frac{X}{U_1}\right]$ and $x_2 \equiv \left[\frac{X}{U_2}\right]$ then, using Rule 4), we can write:

Rule 4.1)

$$x_1 \left[\frac{U_1}{Y}\right] = x_2 \left[\frac{U_2}{Y}\right] \qquad for\ any\ [Y] \tag{7}$$

NOTA BENE ! . . . many a clever person would like to remember this with the rule . . .

$$(new\#)\,[New\,Unit] = (old\#)\,[Old\,Unit]$$

But this is not *quite* true as stated here because . . . well . . . $[New\,Unit]$ is *not* a number and we can't use it in arithmetic ! It does not fall within the realm of our ordinary algebra. Of course . . . logic and reason notwithstanding . . . many of us *secretly* 'remember it' this way and we "get away with it" because it looks so much like Rule 4.1). What we ***really*** mean (but most people don't understand) will be seen and understood much better if we notationally use a third *unit amount symbol* for $[Y]$ (which we may now call $[U_{next}]$) ... then the whole thing "looks like":

Rule 4.2)

$$(new\#)\left[\frac{New\,Unit}{U_{next}}\right] = (old\#)\left[\frac{Old\,Unit}{U_{next}}\right] \tag{8}$$

A concrete example of this would be, for example: $150\ cm\ =\ 1.5\ m$.

It is exactly these unit symbols that you thought were *so familiar* . . . but that we must now understand much more fully. They were actually introduced by the great (!) Scottish physicist James Clark Maxwell as a clever way to keep track of conversion factors. They are actually numbers ... and the idea is <u>so clever</u> that most people don't understand them to this very day! Exercise for the reader: take Rule 4.2) and then use Rule 3) to directly recover Rule 0).
As we will come to understand, a unit symbol has a *hidden partner*. The *cm* symbol used above stands for the comparison between the centimeter unit length and the *next unit* you may wish to choose some day. We don't express that *next unit* explicitly . . . so it is a presence you don't see. It is a *hidden partner*! In explicitly mentioning the name "centimeter" it tells you that the *last* unit used **was** the centimeter. This you knew! But then . . . it has an <u>unnamed</u> function which is to act as a place-holder for the conversion factor you'll need should you someday choose a *new* (*but not **yet** chosen !*) final unit. This trick is so clever . . . and so rarely mentioned . . . that it has fooled many generations of professional scientists. Notice carefully that in the innocent looking expression above $150\ cm\ =\ 1.5\ m$ . . . there are actually **three** units present. There are two that you see . . . and the *hidden partner* of the yet unchosen unit that you don't see. We will summarize all these properties below and you will see that you have little or nothing to learn! The usage will be only what you are used to and familiar with. It is only that you probably didn't <u>understand</u> fully *what* you were doing all those years . . . even though you knew well *how* to do it.

Any number obtained from a comparison process and depending on the choice of some ***arbitrarily chosen and <u>yet</u> unspecified*** unit we will now call a "Dimensional Quantity". Combining dimensional quantities creates further dimensional quantities. For example, velocity is a (length # / time #) and mass density is a (mass # / volume #). We commonly speak of the 'net dimension' of a quantity. What we really *mean* by this locution is the required combination of conversion factors from Rule 0) that we would find necessary to apply in order to find the representation number in some new choice of units. The technical term for all of this is the "Dimensional Exponent" belonging to the quantity. For example, the dimensional exponent of d^2x/dt^2 is $(length)^1(time)^{-2}$. You probably knew this.

III. THEORIES, EQUATIONS, AND CONSTANTS OF PROPORTIONALITY

A theory is a statement of a pattern among measured numbers which we conjecture to be true or nearly so. The preferred mode of expression these days is in terms of *equalities* though previously (with great wisdom) physicists of the past stuck to using *proportionalities*. What's the difference? The point is that physical equations having **any real content** relate differing physical quantities measured out in different (and arbitrary) units. By sticking to proportionalities one sidesteps the whole issue of conversion factors! Customary practice nowadays however is to start from a chosen *fundamental defining equation* (i.e. a statement expressed as an equality ...) - which then, virtually always, introduces a new *constant of proportionality*. This is where yet more confusion enters.

example:

Suppose (like Galileo !), by a long and tedious examination of the manner in which "things fall", we were to finally conjecture that: $z \propto t^2$ ...i.e. that a direct proportionality between the distance fallen and the square of the time elapsed is a good model of Nature. Now if we choose to proceed another step and measure out *length* in meters and *time* in seconds we could actually *measure* the implied constant of proportionality and would arrive at the following *equality*:

$$z = 4.9\, t^2 \qquad (\underline{\text{units}}\text{: length in meters, time in seconds})$$

Of course, it might come to pass that on some later day we were motivated (for whatever reason) to express our equation had we chosen to use other units. How then do we change over to the relationship that would have appeared when using those new and different numbers ... but without going through all those tedious direct measurements again? A really confusing point that will emerge here is that there exists both a 'long hand' and a 'shorthand' way to effect this. For reference purposes I will demonstrate both of these in some detail - they are, of course, completely equivalent and the use of one or the other is merely a matter of taste.

1. *Long-Hand (old fashioned) Active Method of changing units.*

First, we begin by assuming that our equation has been expressed in some specific set of units, and that all constants are, accordingly, simple known real numbers, e.g. as before ...

$$z = 4.9\, t^2 \qquad (\text{ current units: meters, seconds})$$

Now we decide to choose new units...say $[U_L]$ for lengths and $[U_T]$ for times.
Next, we express old measures of variables in terms of new measures:

$$z = \left[\frac{Z}{Meter}\right] = \left[\frac{Z}{U_L}\right]\left[\frac{U_L}{Meter}\right] = z'\left[\frac{U_L}{Meter}\right] \tag{9}$$

$$t = \left[\frac{T}{Sec}\right] = \left[\frac{T}{U_T}\right]\left[\frac{U_T}{Sec}\right] = t'\left[\frac{U_T}{Sec}\right] \tag{10}$$

Now insert these new expressions for the old measures into the old equation at each of their occurrences in the equation, ... finally arriving at:

$$z'\left[\frac{U_L}{Meter}\right] = 4.9\left(t'\left[\frac{U_T}{Sec}\right]\right)^2 \tag{11}$$

Finally, collect all of the accumulated conversion factors next to the old constant of proportionality:

$$z' \;=\; 4.9\,\frac{\left(\left[\frac{U_T}{Sec}\right]\right)^2}{\left[\frac{U_L}{Meter}\right]}\, t'^{\,2} \;=\; 4.9\left(\left[\frac{Meter}{U_L}\right]\right)^1\left(\left[\frac{Sec}{U_T}\right]\right)^{-2} t'^{\,2} \tag{12}$$

In the last step we used the simple properties of linear measures derived above. Our crucial recognition at this point is that the old constant of proportionality times the accumulated set of conversion factors defines the *new* constant of proportionality. In modern condensed notation we write: $z' \;=\; 4.9\frac{m}{s^2}\; t'^{\,2}$.

2. Short-Hand (modern) Passive Method of changing units

Here we start (again) with the recognition that all physical constants of proportionality have their origin in some fundamental defining equation. This defining equation uniquely determines **by its form** the set of conversion factors which the proportionality must ultimately accumulate as we go about changing units (see the 'Active Conversion' just above). This accumulated net 'dimensional exponent' is now traditionally *listed* alongside each physical constant. BUT! (. . . and here comes the sticking point!) what custom has conveniently allowed us to *forget* is that the dimensional symbols "m", "kg", "sec" ...etc. are really just *short-hand notation* for the more cumbersome conversion factor symbols used above (I made them cumbersome on purpose to draw your attention . . .), i.e.:

$$m \equiv \left[\frac{Meter}{U_L}\right] \qquad sec \equiv \left[\frac{Second}{U_T}\right] \qquad kg \equiv \left[\frac{kilogram}{U_M}\right] \ \text{etc.}$$

Now! . . . when we write out an equation, by merely including the dimensional symbols along side our constants, we can **automatically** arrive at what a tedious long-hand changing of units must necessarily have given us also (convince yourselves of this crucial fact!). As we emphasized above, this trick is **so** clever that most people (even scientists !) . . . "don't get it" ! The crucial understanding is, again, that those unit symbols are actually *conversion factors* between the last used (and thus fully "known") set of units and the next to be chosen (but not *yet* chosen . . . so not *yet* known) set of units. In using this method we agree to drag along an undetermined but precisely placed conversion factor symbol (the unit symbol) which stands ready to be used at any moment we might choose. Talk about clever! It may sound cumbersome at first . . . but it beats cold trying to *guess* at the *end* of the calculation which conversion factors you need.

our example again! . . . but this time using a *passive* method.

We start by referring to standard tables or listings where we read (as usual) that the constant of gravitational acceleration is listed as $9.8\, m^{\mathbf{1}}\, s^{\mathbf{-2}}$. So in 'meter' and 'second' units we may immediately write down

$$z = \tfrac{1}{2}\, 9.8\, t^2$$

since in choosing meters and seconds as our "ultimate units" the conversion factors $\{m,\, sec\}$ both take on the value of unity (i.e. they are both the number "1"), and where, in this equation, our symbols mean:

$$z \equiv \left[\frac{Z}{meter}\right] \qquad t \equiv \left[\frac{T}{second}\right]$$

But then, in $[U_L]$ and $[U_T]$ units, we must include the factors $\{m,\, sec\}$ because their numerical values won't be determined until we make definite choices for $\{[U_L]\,$, and $[U_T]\,\}$. Accordingly, we write :

$$z' = \frac{1}{2}\, 9.8\, m^{\,1} sec^{\,-2}\, t'^{\,2}$$

which is precisely where we arrived before, though now we started with the much more "abstract" meanings:

$$z' \equiv \left[\frac{Z}{U_L}\right] \qquad t' \equiv \left[\frac{T}{U_T}\right] \qquad m \equiv \left[\frac{Meter}{U_L}\right] \qquad sec \equiv \left[\frac{Second}{U_T}\right]$$

NOTA BENE: In general usage we often simply omit the primes on z' and t' for brevity or out of carelessness. I include them here explicitly to emphasize that primed and unprimed symbols are representing **different** (though proportional) numbers.

Please pause here to internalize a crucial fact. The span of real-number values that z' will pass through in describing our experiment has been *scaled* by a factor from the values that z passes through. That scaling factor is precisely the number $m \equiv \left[\frac{Meter}{U_L}\right]$. That is, each value $z' \equiv \left[\frac{Z}{U_L}\right] = \left[\frac{Z}{meter}\right]\left[\frac{meter}{U_L}\right] = z\, m$. This may be your first realization that unit conversion is really a scaling process. Scaling relationships will be a **crucial** piece of the information about how the world fits together. The realization that the *ever-so-humble* unit conversion process and the *ever-so-mighty* scaling relationship process are two sides of the same coin is shocking for sure!

A. Discussion

a) Disadvantages of the passive method:

I must already know the dimensional exponent of every constant I employ. If I don't know this I must and (may always!) rely on the active method.

b) Advantages of the passive method:

(a) Philosophical: Since Nature doesn't *force* any particular set of units upon us . . . it is artificial and misleading to make any arbitrary choices (say M.K.S.). Indeed, by *making* the choice you disguise the truth that no such choice is required. The passive method of writing equations provides a method for writing equations in an unbiased manner. Notice, the equation

$$z = 4.9\, m^{1}\, sec^{-2}\, t^{2}$$

cannot truly be said to **be** in any particular set of units...e.g. the conversion factors: m and sec have not yet been specified, i.e. the ultimate set of units in which things are to be expressed has been left open. Merely by carrying along the (as yet unspecified) numbers 'm' and 'sec' in my algebra I am assured of arriving at conclusions which are valid for any set of units I should ultimately choose. This is a proper reflection of the choice Nature gives us.

(b) Algebraic: By allowing physical constants to carry a dimensional exponent we observe that both sides of any equation *must* have the same net dimensional exponent. Thus, we can check our algebra for errors. This is the old "comparing units" trick. However! It is much (much!) more than just the "comparing units" trick ! By introducing how the equation *must scale* . . . we can discern much about how the solution to this equation must scale too. This will give us "scaling laws" (a simple example is Kepler's Third Law) and will allow us to scale our numbers to those that are simplest for numerical computation (i.e. near "1"). This will lead us to the idea of "Natural Units".

IV. NATURAL UNITS

While we have already stated that imposing our 'arbitrary' choice of units on Nature would certainly be artificial and ugly (poor physics !) - nonetheless it remains true that *Nature* suggests 'natural units' for itself. In each separate problem we encounter, there emerge 'natural sizes' in that problem. For example, in the study of planetary motion, using the 'meter' to measure off distance from the sun is absurd . . . the natural unit would surely be the radius of the orbit. Similarly, in studying the spectrum of hydrogen, measuring the mass of the electron in kilograms couldn't be more ridiculous - the electron *is* the central object of study so let *its* mass be the unit of mass. We will notice that in 'Natural Units' all measured numbers, in fact, come out near 'unity' e.g. a number like 2.86 instead of 4.38×10^{44}. For computer work we want all numbers to be roughly between say 0.1 and 10.0 . . . . *This is always possible!* (a neat theorem assures us of this, which we'll prove later). All numerical work and a good bit of algebraic work is enormously simplified by first finding natural units. More importantly though, by using Natural Units the essential physical relationships will stand out much more clearly because all of the arbitrary and distracting 'crud' has been removed from our algebra. Finding natural units is simple and can be effected either via the long-hand 'Active' way or the short-hand 'Passive' way. Understand here that the choice of natural units is not perfectly unique - though all choices will be 'near-unity' multiples of one another (e.g. using either radius, diameter, or circumference to describe a circle is 'natural').

example:

To illustrate the methods, I'll use as my descriptive example equation:

$$\frac{1}{2}\, m \left(\frac{dx}{dt}\right)^{2} + m\, g\, x = E \tag{13}$$

You will recognize that this comes from our falling body problem again.

A. Finding Natural Units: a practical approach

Step 0: **Philosophical Orientation**

Nature doesn't depend on any human choice of units. Therefore, as we shall find, all physical equations will have their elemental expression in "passive" dimension-*full* form or (even better) as proportionalities. That is, they are not "in" *any* particular set of units. Nonetheless, the symbols in our equations *are* real numbers. So that, in principle, we understand that we *have* measured lengths, times, or whatever against some set of reference amounts. The confusing thing is that we don't *have* to tell anybody (even ourselves!) just what the current choices are that have been made! The way we do this strange thing is to carry along 'as-yet-undetermined' conversion factors which will convert us to 'as-yet-unspecified' units. These are (or more precisely 'represent') the "hidden partners" (i.e. the unspecified unit amounts) throughout our algebra. In the problem at hand we have three free dimensions: length, time, mass. Let the symbols $[U_L]$, $[U_T]$, $[U_M]$ represent the definite but 'as-yet' unspecified unit amounts (the hidden partners).

Accordingly, the symbols x, t, and m actually mean:

$$x \equiv \left[\frac{Abstract\ X\ Displacement}{U_L}\right]$$

$$t \equiv \left[\frac{Abstract\ Time\ Displacement}{U_T}\right]$$

$$m \equiv \left[\frac{Abstract\ Mass}{U_M}\right]$$

Step 1: **Identify Your Constants**

All our equations will have descriptive constants embedded in them. These constants are the means by which the 'sizes' in Nature enter our equations just as the algebraic *structure* of the equation expresses those essential physical relationships which we will call "shape" relationships. These constants can be universal numbers (g), numbers specific to the system (m), or even specifics of the given motion (E). Dimensional constants change their numerical values depending on the choice of units employed and we want the important "natural sizes" in our problem to have 'near unity' numerical values. Our present task ultimately comes down to choosing a specific set of units such that the *numerical values* of these constants comes out to be unity (or very nearly unity). At this step, then, we identify our problem's constants (viz. , for this problem, $\{\ g,\ m,$ and $E\}$) .

Step 2: **Introduce a new set of "Natural" Units . . . but which are not yet known!**

We next wish to introduce a 'new' set of units which will ultimately be our 'Natural Units' - but we must do it "formally" (i.e. abstractly) . . . at this step since we don't know which ones we want yet! To do this I first introduce abstract (and as yet unknown) unit amounts for our three fundamental dynamical dimensions; length, time, and mass:

$$[L] \qquad [T] \qquad [M]$$

and then also the new symbols L, T, and M where *these* symbols stand for *true numbers* (not abstract amounts) because the definite (but still unknown) *abstract amounts* will now have been compared to our *hidden partner amounts* and are, consequently, defined by:

$$L \equiv \left[\frac{L}{U_L}\right] \qquad\qquad T \equiv \left[\frac{T}{U_T}\right] \qquad\qquad M \equiv \left[\frac{M}{U_M}\right]$$

Step 3: **Scale the Variable Quantities**
Now ... whenever we encounter the variable numbers x, t, ... we may write by simple algebra:

$$x = \left(\frac{x}{L}\right) L = x'\, L \qquad \text{where we define} \qquad x' \equiv \frac{x}{L}$$

$$t = \left(\frac{t}{T}\right) T = t'\, T \qquad \text{where we define} \qquad t' \equiv \frac{t}{T}$$

It's important to notice that (here, I have taken length as an example)

$$x' \equiv \frac{x}{L} = \frac{\left[\frac{X}{U_L}\right]}{\left[\frac{L}{U_L}\right]} = \left[\frac{X}{L}\right] \qquad (\ldots \text{where we have used } \textbf{Rule 3)}\ )$$

so that, truly, x' *is* the measure of $[X]$ with respect to $[L]$ etc. . Notice especially that the numbers x' and t' are *no longer* "dimensional"! As the comparison of a definite amount to another definite amount ... the *hidden partners* have **dropped out**. We may now take our starting equation and substitute in for x and t in terms of x' and t'.
Notice, as we begin, that "compound expressions" are re-expressed just as one would expect e.g.:

$$\frac{dx}{dt} \rightarrow \frac{dx'L}{dt'T} = \frac{dx'}{dt'}\left(\frac{L}{T}\right)$$

Our starting equation is now re-expressed as:

$$\frac{1}{2}\, m \left(\frac{dx'}{dt'}\right)^2 \left(\frac{L}{T}\right)^2 + m\, g\, x'\, L = E \tag{14}$$

Step 4: **Render the Equation Dimensionless**

Every passively expressed equation has some net dimensionality. The one we are working with has the dimension of energy. This means that its net dimensionality expressed in basic dimensions is $(mass) \cdot (length)^2 \cdot (time)^{-2}$. We now form the unit of our equation's dimension out of our new (and yet to be discovered) L, T, M units and divide the equation by it. So, in the case at hand, we divide the entire equation by $\left(M\, L^2\, T^{-2}\right)$. Once we have done this we will have rendered the equation completely *dimensionless* . It now appears in the following form:

$$\frac{1}{2}\, \frac{m}{M} \left(\frac{dx'}{dt'}\right)^2 + \frac{m}{M}\, \frac{g}{L\, T^{-2}}\, x' = \frac{E}{M\, L^2\, T^{-2}} \tag{15}$$

Step 5: **Identify the Constant "Clusters"**

At this point we observe that we have accumulated collections or "Clusters" of constants.

In our equation I observe three different clusters of constants:

$$\frac{m}{M}\ , \qquad \frac{g}{\left(\frac{L}{T^2}\right)}\ , \qquad \text{and} \qquad \frac{E}{M\, \left(\frac{L}{T}\right)^2}$$

These Clusters **must** be dimensionless since the entire equation now is ... and the variables we're now using $\{x'\,,\, t'\}$ are too ! Accordingly, the clusters are definite numbers and their values will depend on what we choose for L, T, M .

Step 6: **Set the Clusters Equal to Unity...and Discover your Natural Units.**

At this point we now have the freedom to make our constants "***numerically disappear from view***" so to speak . . . simply by specifying that the constant clusters shall take on the simplest of all numerical values . . . namely unity. Once we make this specification, we may then "back-solve" for L, T, M in terms of our problem's constants. In our case there are three units to adjust and also three constant clusters whose ultimate value we wish to specify.

(a) $1 = m/M$ so! $M = m$

(b) $1 = \frac{g\,T^2}{L}$ so! $\frac{L}{T^2} = g$

(c) $1 = \frac{E}{M\left(\frac{L}{T}\right)^2}$ so! . . . (by combining) $L = \frac{E}{m\,g}$

(d) . . . and now combining again . . . $T = \sqrt{L/g} = \sqrt{E/(mg^2)}$

Summary:

We conclude that our equation can be put in the following *simple* form :

$$\frac{1}{2}\left(\frac{dx'}{dt'}\right)^2 + x' = 1 \tag{16}$$

if we choose units:

(a) $M = m$

(b) $L = \frac{E}{m\,g}$

(c) $T = \sqrt{E/(mg^2)}$

Notice that these units are the sizes which enter this problem . . . and so then the problem has determined its own *"Natural"* units. We might say that that the problem has been *self-referenced*. Everything in the problem is compared to (and only to) something else in the problem. This is the way Physics should be. We take a setting and find relationships *within* that setting alone. That this is always possible to achieve is not at all obvious to young students . . . but in fact it is!

We may revert to the original ***arbitrary*** set of units any time we choose merely by reinserting the definitions:

$$x' = \frac{x}{L} = \frac{x}{\frac{E}{m\,g}}$$

$$t' = \frac{t}{T} = \frac{t}{\sqrt{E/(mg^2)}}$$

Generally, one simply reinserts at the *end* of the problem, because there just isn't any benefit in dragging along constants through our algebra. Notice that in **choosing** a definite set of units our equation is no longer passive! The *hidden partners* are gone! We say that our equation is now **Active**. Notice also that our equation (apparently!) no longer obeys dimensional homogeneity. Actually it does because it no longer has any dimension at all. It is fully dimensionless. All equations that are to be evaluated numerically (say on a computer) must be **Active**. Why? Because computers only understand numbers and comparisons to *hidden partners* are not yet definite numbers. What we have simply done is chosen units that are built out of our constants . . . and then tucked those units under our variables where you don't "see them". You probably didn't suspect that this was always possible - but it always is.

B. Physicists Tricky "Short-Cut" to the Same Result.

What we have just discussed might be called the "Engineer's Work-a-Day Method". It's fool-proof and works every time. Physicists, however, like insightful "short cuts" ... and they have several. An equivalent method (but which young students experience as "cheating") is simply to go back to Step 1) and identify the constants and then immediately use **Rule 4.2)** . We need only realize that we may choose our new units $\{L, T, M\}$ so that the *numerical values* of our base constants go to the number "1" in these units. That is we choose natural units such that:

(a)	$m = 1 \cdot M$	Still in Passive Form !
(b)	$g = 1 \cdot \frac{L}{T^2}$	Still in Passive Form !
(c)	$E = 1 \cdot M \left(\frac{L}{T}\right)^2$	Still in Passive Form !

If you look closely you will discern that this is precisely what we achieved in Step 6) above. Now you would proceed to "back solve" just as you did before to discover $\{L, T, M\}$. When you inset these into the equation you can either divide by the dimension of the equation (as before) or pull the "*trickiest trick*" yet by simply declaring that your *hidden partners* **are** these quantities too ... in which case we have, finally: $L = 1, T = 1, M = 1$. Then you get:

(a)	$m = 1$	Now in Active Form !
(b)	$g = 1$	Now in Active Form !
(c)	$E = 1$	Now in Active Form !

In any of these methods you end up with the very same "active equation" as before with no constants appearing and which is ready for numerical evaluation. This will be one of your basic tools throughout your entire scientific life.

V. UNITS COHERENT WITH A FUNDAMENTAL EQUATION

We have said that theories relate numbers obtained from measuring various physical quantities - and that the fundamental equation defining the relationship invariably introduces a constant of proportionality. At this stage of the discussion we must bemoan that perverse historical custom sometimes violates this otherwise rational state of affairs. It is particularly grievous when this occurs in our most fundamental and beloved equations. Let me say here that *Newton* did it right! It was his successors that messed it up. As an example I use the fundamental law of mechanics. Newton realized that: force, mass, length, and time could all be independently measured.

In his fundamental treatise Newton stated (in principle):

$$\vec{F} \propto \frac{d\, m\vec{v}}{d\, t}$$

namely: your measurements of force (in whatever units) will be found proportional to the time rate of change of momentum (in whatever units).

As an equality we may write, of course:

$$\vec{F} = k\, \frac{d\, m\vec{v}}{d\, t}$$

... where k is the constant of proportionality which arises naturally. It depends on the absolute sizes of all the various units employed, as always.

At this point in our discussion, two historical currents collide. The French revolution provided the impetus to standardize everything. In particular the *meter*, the *kilogram*, and then finally the *second* were well defined (the old standard meter and kilogram still remain in Paris). To adopt a unit of force the suggestion was now made - 'why not choose a unit of force such that the constant of proportionality comes out with the simplest numerical value possible ... the numerical value '1'. This was done - and the resulting unit of force was called the **Newton** (what else?) - and there upon the ancient wisdom was forgotten that in some *other* set of units the constant of proportionality might not remain '1'. Since the continental French set the tone for science well into the modern era we have been stuck with the equation $\vec{F} = 1\,\frac{d\,m\vec{v}}{d\,t}$ ever since.

Consequences

Suppose we take as our assumed working equation $\vec{F} = \frac{d\,m\vec{v}}{d\,t}$ (which *is* true if you measure: length in meters, time in seconds, mass in kilograms, and force in Newtons).

Now examine what we arrive at if we change to units $[U_L]$, $[U_T]$, $[U_M]$, and $[U_F]$. Proceeding as before, we may write:

$$\vec{F'}\left[\frac{U_F}{Newton}\right] = \frac{d\,m'\vec{v'}}{d\,t'}\left[\frac{U_M}{Kilogram}\right]^1\left[\frac{U_L}{Meter}\right]^1\left[\frac{U_T}{second}\right]^{-2}$$

Now collect conversion factors all together, yielding:

$$\vec{F'} = \frac{d\,m'\vec{v'}}{d\,t'}\left[\frac{U_F}{Newton}\right]^{-1}\left[\frac{U_M}{Kilogram}\right]^1\left[\frac{U_L}{Meter}\right]^1\left[\frac{U_T}{second}\right]^{-2}$$

Now, for an arbitrary choice of new units, this product of conversion factors will not equal '1'. From a philosophical point of view, this is only natural - yet, by historical president we have somehow become *invested* in this basic equation having nothing but a '1' standing there (yes, it's idiotic but that's the way it is). We now have no choice. Tradition dictates that if three of the units are chosen at will - the fourth (usually the force unit) *must* be chosen so that the above product equals '1'.

This constraint may be written as the simple requirement:

$$1 = \left[\frac{U_F}{Newton}\right]^{-1}\left[\frac{U_M}{Kilogram}\right]^1\left[\frac{U_L}{Meter}\right]^1\left[\frac{U_T}{second}\right]^{-2}$$

On rearrangement it assumes a more familiar form however:

$$\left[\frac{Newton}{U_F}\right] = \left[\frac{Kilogram}{U_M}\right]^1\left[\frac{Meter}{U_L}\right]^1\left[\frac{second}{U_T}\right]^{-2}$$

The shorthand version of this is the old rule:

$$Newton = Kilogram^1\,Meter^1\,second^{-2}$$

The point is this: If we choose to preserve the constant of proportionality at the value of unity (as convention demands) then the choice of the fourth unit is not free. We may only choose among those special sets of units $[U_F]$, $[U_L]$, $[U_T]$, $[U_M]$ such that:

$$\left[\frac{Newton}{U_F}\right] = \left[\frac{Kilogram}{U_M}\right]^1\left[\frac{Meter}{U_L}\right]^1\left[\frac{second}{U_T}\right]^{-2}$$

All selections of the four units which *do* obey this relation are said to be **coherent** with the fundamental starting equation. The M.K.S.N. set of units is one such selection, and there are an infinity of others.

VI. SUMMARY OF WORKING RESULTS

In this summary we collect a set of final useful results that scientists generally carry around in their heads.

1. Active and Passive

Passive Expressions: By the word "Passive" we denote that the quantity at hand is being expressed without a final choice of units. The way we do this is by introducing "as yet unchosen unit comparison amounts" $\{[U_L]\,,\,[U_T]\,,\,[U_M]\}$ (against which we measure things) and *unit symbols* ! Unit symbols actually carry two kinds of information: 1) they tell you what unit was last used, and 2) by their position they indicate the place (and number) of the conversion factors you will need should you choose to change to new units. Passive equations have dimension-full constants and the equations themselves exhibit "*Dimensional Homogeneity*" (the old matching units trick). Every well defined physical problem contains - within its definitions - a complete (or over-complete!) set of "Natural Units". It is always possible to arrange the equation such that the variables are scaled by (i.e. in ratio with) a natural unit. One key outcome, then, is that when we "do algebra" with the symbols of a passively expressed equation to obtain an answer ...then the argument of any function in that answer **must also be dimensionless**.

Active Expressions: By the word "Active" we denote quantities that have been measured out in some definite unit system. The quantities involved, then, are pure numbers and may be entered into a computing machine. Unit symbols "should be gone" because if we have chosen our final units . . . all existing "conversion factors" have been used up. Of course, by *sloppy habitual usage* many a person "hangs onto" unit symbols until the last moment. Then, at that last instant, before entering numbers into a machine, . . . they "throw them away" - or so they think! Actually - they **don't** throw them away! They actually turn them into the number "1" which is our statement that the units we wish to end up in are the units we are already in! At that moment the "*hidden partners*" are *actually* chosen to be the units we are now in . . . and the unit symbols, accordingly, take on the value unity . . . and vanish from view. Actively expressed equations **DO NOT** apparently obey "*Dimensional Homogeneity*" any longer . . . except that they really do . . . because everything in them is now dimensionless (. . . please view equation (16)).

2. Unit Conversion

Active Form: $(new\#) = (old\#)\left[\frac{Old\,Unit}{New\,Unit}\right]$ Which is **Rule 0)**.

Passive Form: $(new\#)\left[\frac{New\,Unit}{U_{next}}\right] = (old\#)\left[\frac{Old\,Unit}{U_{next}}\right]$ Which is **Rule 4.2)**.

Unit symbols disappear in two ways: 1) when we choose the ultimate "hidden partner" (*the heretofore as-yet-unchosen-unit*) to finally (!) **be** something definite, or 2) we take the ratio of two unit symbols and the "hidden partner" units disappear by ratio-cancelation. An example might be $\frac{meter}{cm} = 100$. Or, again,

$$150\;cm \;=\; 150\;\frac{cm}{meter}\,meter \;=\; 150\,(.01)\,meter \;=\; 1.5\;meter$$

3. Natural Units

The "Algebraic Structure" of our defining equations defines what we will call the set of **Shape Relations** in our problem. The constants define the **Size Relations**. (*As a clear example of this, reflect on all the properties of a circle that don't depend on it's size . . . which is most of them! In choosing "Natural Units" it will be like studying the "unit circle")*. Buried within every problem are enough constants (often more than enough !) to provide Natural Units for every quantity. When we do this: 1) the constants "disappear from view", 2) the "shape properties" of our problem are far more easily seen, 3) the numerical values which we find ourselves using will be "near unity" in size and thus "Well Conditioned" (i.e. ideal for machine evaluation).

a) In the simplest Mechanics Problems we have only three base dimensions whose units we may freely choose: $\{$*length, time, mass* $\}$. Accordingly, we may take (at most) any three of our constants and specify that they take on the value of "1" in our new {L, T, M } system of "Natural Units" which we have now defined by that act.

b) Once we have chosen Natural Units and divided out the equation's dimension, our equations will be fully dimensionless and will no longer exhibit "apparent" dimensional homogeneity.

c) All functions representing the solution to any Physical Problem must have *DIMENSIONLESS* arguments.

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