CSUC
Department of Physics
Class Notes

Pre-Introduction to ***Determinants***

I. *INTRODUCTION*

Determinants have their origin in the study of simultaneous linear equations. Very early on, pioneers in mathematics realized that they were repeating the same algebraic maneuvers time and time again and consequently sought to codify the "repeatable steps" of their operations. The outcome, among other things, was the theory of Determinants. Only later, others began to recognized that Determinants are *absolutely central* to the entire study of "Linear Spaces" altogether. Where does this "centrality" come from ? As it will turn out, a Determinant is not just an amazingly handy tool (which it is !) but also the unique "determiner" of whether a set of vectors are, in fact, *linearly independent*. From this fact alone follows almost everything. In addition, since we ascribe to Hermann Weyl's central dictum that all important physical phenomena correspond to significant geometric constructions, we are not - in the least! - surprised that length, area, volume, hypervolume etc. are all best expressed in terms of Determinants. This *Pre-Introduction* provides a gentle entry point into the discussion. Modern students have the benefit of being pretty familiar with ordinary spatial 3-vector operations - so they have been playing with Determinants all along whether they knew it or not! The ordinary vector *dot* and *cross* product operations are all we need.

II. *DETERMINANTS IN GEOMETRIC VECTOR SPACES*

First, a determinant is a **number** extracted from a set of vectors (very very much like an ordinary "dot-product"). We will need as many vectors as the size of the space - so for solid geometry we will need three 3-vectors. Perhaps you are confused ... *already*! After all, in solving sets of linear equations we don't seem to have *any* "vectors" in the discussion at all and isn't that where determinants have their origin to begin with? ... OK! OK! Let me start again. A determinant is a **number** extracted in a definite (pretty simple) algorithmic way from any *square array* of other numbers. The match-up between the two ideas is that, vectors can always be represented by their components along a set of basis vectors. Thus three 3-vectors can be known by the three sets of three-components-each that can then be lined up next to each other and I will now have my required 3 × 3 square array. The reason I didn't say this at the outset is that the "value" of the numerical-output that the determinant gives me does NOT depend on which particular basis I might have chosen ! This is exactly analogous to the fact that ordinary "dot-products" don't depend on the basis you use to compute them either. All significant things are like this - nothing arbitrary matters. So! I don't have to tell you which basis I might be choosing today ... because it doesn't affect anything. In this way we are brought to the mind-frame of modern mathematics. Only talk about those things that matter. A vector itself is *not* "just a number" (in 3-space we say it has "direction"). It can be viewed as an abstract thing (which yes... *interacts* with ordinary numbers ... but isn't itself one ...) and having, rather, it's own standing in the world. So I really am justified in thinking of the determinant as a function from the three vectors to the *Real Numbers*. This output for our system of 3-D spatial vectors is, in fact, also known as the vector triple product of the vectors! Specifically, if we are given geometric 3-vectors $\{\vec{a}\,,\,\vec{b}\,,\,\vec{c}\}$, then we *define* :

$$det\left(\vec{a}\,,\,\vec{b}\,,\,\vec{c}\right) \equiv \vec{a}\cdot(\vec{b}\times\vec{c}) = \vec{b}\cdot(\vec{c}\times\vec{a}) = \vec{c}\cdot(\vec{a}\times\vec{b})$$

I probably should have mentioned that one can just as easily define the determinant for vectors confined to a *plane* too. Here, however the "dimension of the space" is only two and I will need just two 2-vectors. Suppose we have, in this case, the two vectors: $\vec{a} \equiv (\,a_1\,,\,a_2\,)$ and $\vec{b} \equiv (\,b_1\,,\,b_2\,)$. In this case, the determinant is probably most familiar as a simple algebraic combination of the components:

$$det\left(\vec{a}\,,\,\vec{b}\right) \equiv \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \equiv (a_1\,b_2 - a_2\,b_1\,)$$

Here too, the output value does not depend on the choice of basis vectors along which we evaluated components. For those more familiar with cross products it may be known that *each* of the components of the cross product of two 3-vectors is exactly just such a 2×2 determinant of specific components, viz.

$$\left(\vec{a} \times \vec{b}\right) \equiv \hat{x} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \hat{y} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \hat{z} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

This sum is often, then, itself represented and remembered as a single 3×3 square array:

$$\left(\vec{a} \times \vec{b}\right) \equiv \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

A good many people think of determinants exclusively in terms of the manipulations involving these $N \times N$ square arrays of numbers. This is, indeed, where determinants had their origins, although we now view them in a much more fundamental way as a "function" from a set of "abstract" vectors to the real numbers. Most students are willing to view *dot products* in this manner and it is a short jump to including determinants in this perspective also.

At this point we should probably list the specific features here that *all* determinants will *always* have and which really are the definitive features of the determinant function. One feature of our discussion that may require some "getting used to" is that each statement can be represented in the more abstract "whole vector" way or in the "more concrete" component way. You should realize that both manners of expression are completely equivalent and choosing one over the other is, to some extent, a matter of style and taste. You will learn to appreciate the powers of both. Here, then, are the defining properties of our determinants.

- **Properties:**

  1. First, we specify that the interchange of *any* two vectors introduces an overall minus sign: e.g.

  $$det\left(\vec{a}\,,\,\vec{b}\,,\,\vec{c}\right) = -\,det\left(\vec{b}\,,\,\vec{a}\,,\,\vec{c}\right)$$

  which in our "triple-product" setting is just e.g. ...

  $$\vec{a} \cdot (\vec{b} \times \vec{c}) = -\vec{b} \cdot (\vec{a} \times \vec{c})$$

  (... plus all the other permutations on this theme ...)

  2. Second, we specify that the determinant shall be linear in each vector entry. That is:

  $$det\left(\alpha\,\vec{a}_1 + \beta\,\vec{a}_2\,,\,\vec{b}\,,\,\vec{c}\right) = \alpha\, det\left(\vec{a}_1\,,\,\vec{b}\,,\,\vec{c}\right) + \beta\, det\left(\vec{a}_2\,,\,\vec{b}\,,\,\vec{c}\right)$$

  for any two numbers α and β and vectors $\vec{a}_1$ and $\vec{a}_2$. This property is self evident for the triple product.

  3. Finally, notice that $\vec{a} \cdot (\vec{b} \times \vec{c})$ yields the volume of the parallelepiped defined by having $\{\,\vec{a}\,,\,\vec{b}\,,\,\vec{c}\}$ all emerge from one vertex. This volume is non-zero if and only if $\{\,\vec{a}\,,\,\vec{b}\,,\,\vec{c}\}$ are linearly independent and, accordingly, span three-space. If these three vectors happen to be the orthogonal right-handed basis vectors $\{\,\hat{x}\,,\,\hat{y}\,,\,\hat{z}\,\}$, then we demand that our determinant shall yield the volume of unity.

  $$det\,(\,\hat{x}\,,\,\hat{y}\,,\,\hat{z}\,) = 1$$

  In our more formal notation we demand, then:

  $$det\,(\,\hat{e}_1\,,\,\hat{e}_2\,,\,\hat{e}_3\,) = 1$$

  That is, the determinant of *any* triad of orthonormal right-handed basis vectors (including, especially the standard one) yields unity. It is *not* entirely obvious, perhaps, that these three specification *alone (!)* completely determine what value a determinant gives. But they do!

At this point it is perhaps appropriate to pause and reflect. We have said elsewhere that the key ideas underlying all linear spaces (i.e. vector spaces) are the concepts of:

- ***linearity***
- ***independence***

This is all fine, but you may justifiably ask just *how* are we going to determine whether any two or any larger set of vectors really are independent ? Start with just two. When are two vectors "the same" or in fact when are *any* two things the same? A good solution to this problem (both mathematically and *logically*) is to say that two things are the same if, when we interchange them ... nothing changes! This is where the determinant comes in. Because, when you interchange any two vectors in the determinant the over all *sign* changes (always !). But if the vectors are the same it can't change! The only number that doesn't change when you change its sign is the number zero. Zero is unique among all numbers by having this property and consequently being the only one without an *inverse*. So the determinant will tell us if any two vectors are in the same "direction" i.e. linearly dependent. Since the determinant is linear in each entry, the overall size is not the important thing - only the overall "direction". What we are *really* testing with Property 1.), then, is whether or not two vectors have the same "direction"! You may also think of this in terms of geometry. Any two "geometric-vectors" pointing in the same direction contain "no volume". The fancy way to give a name to this property is to say that a determinant is "totally anti-symmetric" and this anti-symmetry is where the determination of identity is made - no two vectors may be scalar multiples of each other without the determinant yielding that unique number zero!

Property 1.) in conjunction with property 2.) assures us further that if any vector is even just a *linear combination* of the remaining vectors in the determinant, that then the output will also be zero. To demonstrate this suppose, by way of example, that $\vec{a} = \alpha \vec{b} + \beta \vec{c}$. Then:

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = (\alpha \vec{b} + \beta \vec{c}) \cdot (\vec{b} \times \vec{c}) = \alpha \vec{b} \cdot (\vec{b} \times \vec{c}) + \beta \vec{c} \cdot (\vec{b} \times \vec{c}) = 0$$

since each final term has a repeated vector.

Property 3.) is an overall "normalizing" specification. It sets a specific *size* (namely *unity*) against which all other determinants are to be compared.

III. *APPLICATIONS*

Once we grasp the idea of determinants many applications (some rather surprising ...) spring to mind. Here is a tiny sampling.

A. The Reciprocal Basis

Suppose our problem contains three linearly independent vectors viz. $\{ \vec{a}_1 , \vec{a}_2 , \vec{a}_3 \}$ which we would like to use to define a "basis". In general, these vectors won't be orthogonal or normalized and this can be a real nuisance! You could go through the process of generating an equivalent orthonormal set from them ... but that's work! It's much smarter to realize that one can easily generate a second set of three vectors that we might as well call $\{ \vec{A}_1 , \vec{A}_2 , \vec{A}_3 \}$ that are "orthonormal" not against *themselves* but against the original set! For notational convenience, let's first define the number $D \equiv det(\vec{a}_1 , \vec{a}_2 , \vec{a}_3) = \vec{a}_1 \cdot (\vec{a}_2 \times \vec{a}_3)$ which by assumption is *not* zero. We then proceed to define our new vector set by the following prescription:

$$\begin{aligned}
\vec{A}_1 &\equiv \frac{1}{D} \vec{a}_2 \times \vec{a}_3 \\
\vec{A}_2 &\equiv \frac{1}{D} \vec{a}_3 \times \vec{a}_1 \\
\vec{A}_3 &\equiv \frac{1}{D} \vec{a}_1 \times \vec{a}_2
\end{aligned}$$

Notice that $\vec{A}_1$ must be perpendicular to $\{\,\vec{a}_2\,,\,\vec{a}_3\,\}$ whereas we have $\vec{A}_1\,\cdot\vec{a}_1\;=\;1$ by explicit construction. So each of the $\vec{A}_i$ has unit projection against its own $\vec{a}_i$ but is perpendicular to the other two. We can summarize what we have created by writing:

$$\vec{A}_i\,\cdot\,\vec{a}_j\;=\;\delta_{i\,j}$$

Imagine now the following very common problem. Suppose we need to find a vector $\vec{v}$ that is specified by the requirement that it satisfy the following dot product equations:

$$\begin{aligned}\vec{v}\cdot\vec{a}_1 &= d_1\\ \vec{v}\cdot\vec{a}_2 &= d_2\\ \vec{v}\cdot\vec{a}_3 &= d_3\end{aligned}$$

for some given set of constants $\{\,d_1\,,\,d_2\,,\,d_3\,\}$. The solution is almost trivial and may be found by inspection!

$$\vec{v}\;=\;d_1\,\vec{A}_1\;+\;d_2\,\vec{A}_2\;+\;d_3\,\vec{A}_3$$

Convince yourself that this combination does indeed satisfy the required equations! Therefore it solves the problem (and is, in fact, unique!) . This new set $\{\,\vec{A}_1\,,\,\vec{A}_2\,,\,\vec{A}_3\,\}$ is called the “Reciprocal Basis” associated with the original set $\{\,\vec{a}_1\,,\,\vec{a}_2\,,\,\vec{a}_3\,\}$ ... and basically *solves* most linear algebraic problems.

B. Area as a Vector

Any two displacement vectors, say $\{\,\vec{A}_1\,,\,\vec{A}_2\,\}$, that are not co-linear define a parallelogram of area that should be considered directional ... after all it has an orientation specified by its normal - the common perpendicular. In point of fact, the cross product $\vec{A}_1\,\times\,\vec{A}_2$ *is* the (only!) natural and complete way to talk about the contained area! But suppose we had another parallelogram of area, say $\vec{B}_1\,\times\,\vec{B}_2$ ... can we talk about projecting one area on another ? Can we take a dot product between areas? Can we do all the things with this “vector” that we did with our original vectors? The answer to all these questions is a resounding YES! What we’ve really done is create a natural *second* vector space out of the first. Assertion: the natural dot product between two areas is a ... you guessed it ... a determinant! Let me just show you how it works.

$$(\vec{A}_1\,\times\,\vec{A}_2)\,\cdot\,(\vec{B}_1\,\times\,\vec{B}_2)\;=\;\begin{vmatrix}(\vec{A}_1\cdot\vec{B}_1) & (\vec{A}_1\cdot\vec{B}_2)\\ (\vec{A}_2\cdot\vec{B}_1) & (\vec{A}_2\cdot\vec{B}_2)\end{vmatrix}\;=\;(\vec{A}_1\cdot\vec{B}_1)(\vec{A}_1\cdot\vec{B}_2)\;-\;(\vec{A}_2\cdot\vec{B}_1)(\vec{A}_2\cdot\vec{B}_2)$$

So the “new” dot product is formed out of a 2×2 determinant of “old” dot products (you may think of it like one component of a cross product). The “ij” element of the 2×2 square array is just $\vec{A}_i\cdot\vec{B}_j$. Verify for yourself that another approach using standard “old fashioned” vector identities gives the *same* answer but not the same insight! I think you can surmise where this might be going.

C. Expansions in Components

If we let the set $\{\hat{e}_1,\,\hat{e}_2,\,\hat{e}_3\}$ represent the three axis directions of a right-handed orthonormal coordinate basis where $\hat{e}_i\cdot\hat{e}_j=\delta_{i\,j}$ then, any and all vectors may be expanded in components along these directions and this is how we will generate component expressions out of abstract “whole-vector” expressions as for example, our dot product now becomes:

$$\begin{aligned}\vec{A} &= \sum_{i=1}^{3}\hat{e}_i\,\hat{e}_i\cdot\vec{A}\;=\;\sum_{i=1}^{3}\hat{e}_i\,A_i \qquad (1)\\ \vec{B} &= \sum_{j=1}^{3}\hat{e}_j\,\hat{e}_j\cdot\vec{B}\;=\;\sum_{j=1}^{3}\hat{e}_j\,B_j \qquad \textit{and therefore...}\\ \vec{A}\cdot\vec{B} &= \left(\sum_{i=1}^{3}\hat{e}_i\,A_i\right)\cdot\left(\sum_{j=1}^{3}\hat{e}_j\,B_j\right)\;=\;\sum_{i=1}^{3}\sum_{j=1}^{3}A_iB_j\,\hat{e}_i\cdot\hat{e}_j\;=\;\sum_{i=1}^{3}A_iB_i\end{aligned}$$

In this last result we have used the useful notational result $\sum_{j=1}^{3} \delta_{i\,j}\, B_j \;=\; B_i$.

Now we proceed to define the component expression of the vector triple product very much as we did with our dot product . First we produce the ordinary cross product:

$$\vec{A} \;=\; \sum_{j=1}^{3} \hat{e}_j\, A_j \quad and \quad \vec{B} \;=\; \sum_{k=1}^{3} \hat{e}_k\, B_k \quad and\; therefore...$$

$$\vec{A} \times \vec{B} \;=\; \left(\sum_{j=1}^{3} \hat{e}_j\, A_i \right) \times \left(\sum_{k=1}^{3} \hat{e}_k\, B_k \right) \;=\; \sum_{j=1}^{3}\sum_{k=1}^{3} A_j B_k\, \hat{e}_j \times \hat{e}_k \;=\; \sum_{j,k=1}^{3} A_j B_k\, \hat{e}_j \times \hat{e}_k$$

Since the Cartesian approach is committed to expressing all vector expressions in their component forms, we complete the derivation by finding a *typical component* of $\vec{A} \times \vec{B}$. We recall that, in all cases, the i^{th} component of any vector $\vec{V}$ is found by computing the projection of that vector along the i^{th} coordinate direction by means of the dot product. That is, $V_i = \hat{e}_i \,\cdot\, \vec{V}$. It then follows that:

$$(\vec{A} \times \vec{B})_i \;= \hat{e}_i \cdot (\vec{A} \times \vec{B}) \;=\; \hat{e}_i \cdot \sum_{j,k=1}^{3} A_j B_k\, \hat{e}_j \times \hat{e}_k \;=\; \sum_{j,k=1}^{3} A_j B_k\, \hat{e}_i \cdot (\hat{e}_j \times \hat{e}_k) \tag{2}$$

Notice especially that with two separate indices each sweeping through three values, that there are *nine* terms in this double sum ... yet all but two of them turn out to be zero! This is because of the curious vector triple product term $\hat{e}_i \cdot (\hat{e}_j \times \hat{e}_k)$ in the argument. You can easily convince yourself that to be anything but zero, the indices $\{i, j, k\}$ must represent the numbers $\{1, 2, 3\}$ in some order. That is, if any two of the indices represent the *same* number, then the triple product comes out to be zero. Finally, if $\{i, j, k\}$ represent $\{1, 2, 3\}$ in any*cyclic* (sequential) order i.e. $1 \rightarrow 2 \rightarrow 3 \rightarrow 1 \ldots$ etc. , then the value is 1 and in any *anti*-cyclic order it comes out -1. Because we will be doing so many component calculations we conventionally use the concise notation $\epsilon_{ijk} \;\equiv\; \hat{e}_i \cdot (\hat{e}_j \times \hat{e}_k)$. This symbol (frequently called the *Levi-Civita totally antisymmetric tensor*) embodies the component expression of the cross product, and we simply remember:

$$(\vec{A} \times \vec{B})_i \;=\; \sum_{j,k=1}^{3} \epsilon_{ijk} A_j B_k \tag{3}$$

We summarize the properties of this odd but extremely useful device in the following list:

1. $\epsilon_{ijk} = -\epsilon_{jik}$ i.e. the value changes by a *minus sign* under interchange of *any two* of its indices.
2. $\epsilon_{ijk} = \epsilon_{jki} = \epsilon_{kij}$ i.e. the value is *unchanged* under any cyclic rotation *of all the indices.*
3. $\sum_{k=1}^{3} \epsilon_{ijk}\, \epsilon_{kst} \;=\; \sum_{k=1}^{3} \epsilon_{kij}\, \epsilon_{kst} \;=\; \sum_{k=1}^{3} \hat{e}_k \cdot (\hat{e}_i \times \hat{e}_j)\; \hat{e}_k \cdot (\hat{e}_s \times \hat{e}_t) \;=\; (\hat{e}_i \times \hat{e}_j) \,\cdot\, (\hat{e}_s \times \hat{e}_t) \;=\; \delta_{is}\delta_{jt} - \delta_{it}\delta_{js}$

This third property may take some staring at ... but actually follows straight forwardly enough from our discussion of the dot product of two "areas". We will need this result to evaluate expressions involving two cross products.

Finally, we complete the component expression of the vector triple product with the following sequence of equations:

$$\vec{A} \,\cdot\, (\vec{B} \times \vec{C}) \;=\; \sum_{i=1}^{3} A_i\, (\vec{B} \times \vec{C})_i \;=\; \sum_{i,j,k=1}^{3} \epsilon_{ijk} A_i\, B_j\, C_k$$

So this formula gives us the component expression for the determinant.

$$det\left(\vec{A},\, \vec{B}\,,\, \vec{C}\right) \;=\; \sum_{i,j,k=1}^{3} \epsilon_{ijk} A_i\, B_j\, C_k \;\equiv\; \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

$$=\; A_1(B_2C_3 \,-\, B_3C_2) \,+\, A_2(B_3C_1 \,-\, B_1C_3) \,+\, A_3(B_1C_2 \,-\, B_2C_1)$$

This is the "starting point" for finding the determinant from all those square-arrays of numbers. It connects an expression involving "abstract" vectors (on the "left-hand-side") to one involving explicit components, i.e. "numbers" (on the "right-hand-side"). The "magic" is all in the odd symbol $\epsilon_{ijk} \equiv \hat{e}_i \cdot (\hat{e}_j \times \hat{e}_k)$ that expresses the numerical out-put from the vector triple-product of any three unit basis vectors. Apparently, if we know what happens to basis vectors ... and if we retain linearity ... then we know everything! The miracle of modern mathematics is recognizing just how much structure can emerge from such a *tiny* starting place! We will find that big determinants can be reduced to sums of yet "littler" determinants ... ad infinitum ... and an *enormous* amount of significant structure and consequence follows.

Exercise for the reader!

Suppose that we are given a set of vectors $\{ \vec{a}_1 , \vec{a}_2 , \vec{a}_3 \}$ and define :

$$D \equiv det\,(\vec{a}_1 , \vec{a}_2 , \vec{a}_3)$$

Now form the "Reciprocal Basis" $\{ \vec{A}_1 , \vec{A}_2 , \vec{A}_3 \}$ and take the determinant of *them* ! Show that:

$$det\left(\vec{A}_1 , \vec{A}_2 , \vec{A}_3\right) = \frac{1}{D}$$

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