Causal Response of a S. H. Oscillator

We suppose that we have a traditional simple harmonic oscillator described by Newton's 2^{nd} laws. We now add an external driving force $F_d(t)$ which we suppose is turned on at time t_0. If the oscillator was entirely at rest until that moment, it now responds "causally" and starts to move in response to the stimulus.

Our descriptive picture is, then:

1) $$m\ddot{x} = -kx - b\dot{x} + F_{drive}(t)$$

where $x(t) = \dot{x}(t) \equiv 0$ for $t < t_0$.

We now divide by the mass "m" and define our constant clusters traditionally as follows:

$$\frac{k}{m} \equiv \omega_0^2 \quad ; \quad \frac{b}{m} \equiv 2\beta \quad , \quad \frac{F_d}{m} \equiv f_d(t)$$

and achieve

2) $$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f_d(t)$$

Next we "factor" the left hand side using the quadratic "roots" $\omega_{1,2} = -\beta \pm \sqrt{\beta^2 - \omega_0^2}$

3) $$\left(\frac{d}{dt} - \omega_1\right)\left(\frac{d}{dt} - \omega_2\right) x(t) = f_d(t) \quad .$$ where ...

$$\omega_1 \omega_2 = \omega_0^2 \quad \text{and} \quad \omega_1 + \omega_2 = -2\beta$$

Our task is to "remove" the left-hand side factors by integrating them off. We use the following standard "tricks" repeatedly:

a) $\left(\frac{d}{dt} - \omega\right) \equiv e^{\omega t} \frac{d}{dt} e^{-\omega t}$

b) if $G(t) \equiv \int_{t}^{t} dt'\, g(t')$ then $\frac{d}{dt} G(t) = g(t)$

so that if $H(t) \equiv \int_{t_0}^{t} dt'\, h(t')$ we may write

the differential expression $dH(t) = dt\, h(t)$ and so

$$dH(t)\, G(t) = d\left(H(t)\, G(t)\right) - H(t)\, dG(t)$$

which is just integration by parts in differential form.

Here we go! Start from equation 3)

4) $\left(e^{\omega_1 t} \frac{d}{dt} e^{-\omega_1 t}\right)\left(e^{\omega_2 t} \frac{d}{dt} e^{-\omega_2 t}\right) x = f_d(t)$

4a) $\frac{d}{dt}\left\{ e^{(\omega_2 - \omega_1)t} \frac{d}{dt} e^{-\omega_2 t} x \right\} = e^{-\omega_1 t} f_d(t)$

now integrate from $t_0 \to t$, use $x(t_0) = \dot{x}(t_0) = 0$

$$4_b)\quad e^{(\omega_2-\omega_1)t}\frac{d}{dt}\left(e^{-\omega_2 t}x\right) = \int_{t_0}^{t} e^{-\omega_1 t'} f_d(t')\,dt'$$

$$4_c)\quad \frac{d}{dt}\left(e^{-\omega_2 t}x\right) = e^{-(\omega_2-\omega_1)t}\int_{t_0}^{t} e^{-\omega_1 t'} f_d(t')\,dt'$$

Now multiply by dt and use trick b)

$$4_d)\quad dt\,\frac{d}{dt}\left(e^{-\omega_2 t}x\right) = \underbrace{dt\,e^{-(\omega_2-\omega_1)t}}_{dH}\ \underbrace{\int_{t_0}^{t} e^{-\omega_1 t'} f_d(t')\,dt'}_{G(t)}$$

$$= d\left\{\frac{e^{(\omega_1-\omega_2)t}}{\omega_1-\omega_2}\int_{t_0}^{t} e^{-\omega_1 t'} f(t')\,dt'\right\} - \frac{e^{(\omega_1-\omega_2)t}}{\omega_1-\omega_2}\,e^{-\omega_1 t} f_d(t)\,dt$$

Now integrate again: $t_0 \to t$

$$4_e)\quad e^{-\omega_2 t}x(t) = \frac{e^{(\omega_1-\omega_2)t}}{\omega_1-\omega_2}\int_{t_0}^{t} e^{-\omega_1 t'} f_d(t')\,dt' - \int_{t_0}^{t}\frac{e^{-\omega_2 t'}}{\omega_1-\omega_2} f_d(t')\,dt'$$

$$4_f)\quad x(t) = \int_{t_0}^{t}\left(\frac{e^{\omega_1(t-t')} - e^{\omega_2(t-t')}}{\omega_1-\omega_2}\right) f_d(t')\,dt'$$

and we are done ... except for the discussion!

Observe what we have achieved!

$$x(t) = \int_{t_0}^{t} G(t,t')\, f_d(t')\, dt'$$

$$\text{where } G(t,t') \equiv \frac{e^{\omega_1(t-t')} - e^{\omega_2(t-t')}}{\omega_1 - \omega_2}$$

This very special function is called the "causal Green's function" ... or ... the "Propagator".

It "propagates" the disturbance forward in time.

Since $G(t,t')$ is built out of the homogeneous solutions of the S.H.O. equation ... it actually <u>is</u> a solution of the homogeneous equation. It has the further properties that:

i) $G(t,t') \to 0$ whenever $t = t'$

ii) $\frac{\partial}{\partial t} G(t,t') \to 1$ if $t = t'$

iii) $$\frac{d^2}{dt^2} G(t,t') + 2\beta \frac{d}{dt} G(t,t') + \omega_0^2 G(t,t') = 0$$

for all t'. (i.e. it's a solution of 2))

We can now check:

if $\quad x(t) \equiv \int_{t_0}^{t} G(t,t')\, f_d(t')\, dt' \quad$ then

1st) $\quad \dot{x}(t) = \int_{t_0}^{t} \frac{\partial G}{\partial t}(t,t')\, f_d(t')\, dt'$

2nd) $\quad \ddot{x}(t) = f_d(t) + \int_{t_0}^{t} \frac{\partial^2 G}{\partial t^2}(t,t')\, f_d(t')\, dt'$

3rd] $\quad \ddot{x} + 2\beta \dot{x} + \omega_0^2 x = f_d(t)$

If, in addition, we express $\quad \omega_1 = -\beta + i\sqrt{\omega_0^2 - \beta^2}$

$$\omega_2 = -\beta - i\sqrt{\omega_0^2 - \beta^2}$$

then

$$G(t,t') = e^{-\beta(t-t')} \frac{\sin\left(\sqrt{\omega_0^2 - \beta^2}\,(t-t')\right)}{\sqrt{\omega_0^2 - \beta^2}}$$

a purely <u>real</u> function.

We may think of $G(t,t')$ as the "inversion" or "reciprocal" of our original differential operator.