Simple Harmonic Oscillation:

Steady-State Solution to a Harmonic Drive.

Start: $m\ddot{x} + b\dot{x} + kx = F_0 \cos(\omega t)$ ... divide by m:

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = \frac{F_0}{m}\cos(\omega t)$$

where $\frac{b}{m} \equiv 2\beta = \frac{1}{\tau} = \frac{\omega_0}{Q}$; $\frac{k}{m} = \omega_0^2$.

Trick... let $\begin{cases} \cos(\omega t) = Re[e^{-i\omega t}] \\ x(t) = Re[x + iy] \equiv Re[z] \end{cases}$

Trick: $\frac{d^2}{dt^2} + 2\beta\frac{d}{dt} + \omega_0^2 = \left(\frac{d}{dt} - \omega_+\right)\left(\frac{d}{dt} - \omega_-\right)$

where we have factored $D^2 + 2\beta D + \omega_0^2$

$$\omega_\pm = -\beta \pm i\sqrt{\omega_0^2 - \beta^2}$$

and so $\omega_+\omega_- = \omega_0^2$

$$\omega_+ + \omega_- = -2\beta = -\omega_0/Q$$

Essential Fact: If the drive is $F_0 e^{-i\omega t}$, then the steady state solution is $z(t) = Ae^{-i\omega t}$.

"A" is complex and carries the magnitude & phase.

To obey the dynamical equation, z satisfies:

$$\left(\frac{d}{dt} - \omega_+\right)\left(\frac{d}{dt} - \omega_-\right) A e^{-i\omega t} = \frac{F_0}{m} e^{-i\omega t}$$

$$(-i\omega - \omega_+)(-i\omega - \omega_-) A e^{-i\omega t} = \frac{F_0}{m} e^{-i\omega t}$$

$$\left[-\omega^2 + \omega_0^2 + i\omega(\omega_+ + \omega_-)\right] A = \frac{F_0}{m}$$

$$\left[(\omega_0^2 - \omega^2) - i\,\omega\,\frac{\omega_0}{Q}\right] A = \frac{F_0}{m} \quad \text{... divide by } \omega_0^2\text{!}$$

$$\left[1 - \left(\frac{\omega}{\omega_0}\right)^2 - i\,\frac{\omega/\omega_0}{Q}\right] A = \frac{F_0}{m\omega_0^2}$$

call $\omega/\omega_0 \longleftrightarrow \bar{\omega}$ $\Rightarrow$

$$\left[(1-\bar{\omega}^2) - i\,\frac{\bar{\omega}}{Q}\right] A = \frac{F_0}{m\omega_0^2}$$

Now use universal manipulation:

$$a + ib = \sqrt{a^2+b^2}\left(\frac{a}{\sqrt{a^2+b^2}} + i\,\frac{b}{\sqrt{a^2+b^2}}\right)$$

$$= \sqrt{a^2+b^2}\,(\cos\varphi + i\sin\varphi)$$

$$= \sqrt{a^2+b^2}\, e^{i\varphi} \quad \text{where } \frac{b}{a} = \frac{\sin\varphi}{\cos\varphi} = \tan\varphi$$

$$(1-\bar{\omega}^2) - i\,\frac{\bar{\omega}}{Q} = \sqrt{(1-\bar{\omega}^2)^2 + \left(\frac{\bar{\omega}}{Q}\right)^2}\; e^{-i\varphi} \;;\; \tan\varphi = \frac{\bar{\omega}/Q}{1-\bar{\omega}^2}$$

So! $A = \dfrac{1}{1-\bar{\omega}^2 - i\,\bar{\omega}/Q}\;\dfrac{F_0}{m\omega_0^2} = \dfrac{F_0/m\omega_0^2}{\sqrt{(1-\bar{\omega}^2)^2+\left(\frac{\bar{\omega}}{Q}\right)^2}}\, e^{i\varphi}$

$$z(t) = A\,e^{-i\omega t} = \frac{F_0}{m\omega_0^2}\,\frac{e^{-i(\omega t-\varphi)}}{\sqrt{(1-\bar{\omega}^2)^2+\left(\frac{\bar{\omega}}{Q}\right)^2}}\;;\quad \tan\varphi = \frac{\bar{\omega}/Q}{1-\bar{\omega}^2}$$

$$x(t) = \mathrm{Re}[z] = \frac{F_0}{m\omega_0^2}\,\frac{\cos(\omega t-\varphi)}{\sqrt{(1-\bar{\omega}^2)^2+\left(\frac{\bar{\omega}}{Q}\right)^2}}\;;\quad \text{Done!}$$

<u>Notice!</u> $\bar{\omega} \to 0 \Rightarrow \tan\varphi \to 0\;;\; \varphi \to 0$

$\bar{\omega} \to 1 \Rightarrow \tan\varphi \to \infty\;:\; \varphi \leadsto \pi/2$

$\bar{\omega} \to\gg 1 \Rightarrow \tan\varphi \to 0$, ε is <u>negative</u>

$\Rightarrow \varphi \leadsto \pi$

<u>Energy</u>: Start at the top!

$$m\ddot{x} + m\frac{\omega_0}{Q}\dot{x} + kx = F(t) \quad \text{multiply by } \dot{x}\,!$$

$$\frac{d}{dt}\left(\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2\right) = -m\frac{\omega_0}{Q}\dot{x}^2 + \dot{x}F(t)$$

... but this is <u>steady</u> state: average this equation!

$$\lim_{T \to \text{large}} \frac{1}{T}\int_0^T dt\left(\frac{d}{dt}[\text{energy}]\right) \rightsquigarrow \text{zero!}$$

$$\Longrightarrow \quad 0 = \underbrace{\left\langle -m\frac{\omega_0}{Q}\dot{x}^2\right\rangle}_{\text{average power out}} + \underbrace{\left\langle \dot{x}F(t)\right\rangle}_{\text{average power in.}}$$

Observe! $x(t) = x_0(\omega)\cos(\omega t - \varphi)$

$$\dot{x}(t) = -\omega x_0(\omega)\sin(\omega t - \varphi)$$

so! $\langle x^2\rangle = x_0^2\,\frac{1}{2}$

$$\langle \dot{x}^2\rangle = \omega^2 x_0^2\,\frac{1}{2}$$

$$\overline{PE} = \left\langle \tfrac{1}{2}kx^2\right\rangle = \tfrac{1}{2}k\,x_0^2\,\tfrac{1}{2} = \tfrac{1}{4}\frac{k}{m}\,m x_0^2 = \tfrac{1}{4}\omega_0^2\,m x_0^2$$

$$\overline{KE} = \left\langle \tfrac{1}{2}m\dot{x}^2\right\rangle = \tfrac{1}{2}m\,\omega^2 x_0^2\,\tfrac{1}{2} = \tfrac{1}{4}m\,\omega^2 x_0^2$$

so $\dfrac{\overline{KE}}{\overline{PE}} = \left(\dfrac{\omega}{\omega_0}\right)^2 = \bar{\omega}^2$ and ...

$$\bar{U} = \overline{KE} + \overline{PE} = (1+\bar{\omega}^2)\,\overline{PE}$$

so ...

$$\overline{PE} = \frac{1}{1+\bar{\omega}^2}\,\bar{U} \quad \text{\&} \quad \overline{KE} = \frac{\bar{\omega}^2}{1+\bar{\omega}^2}\,\bar{U}$$

these are <u>not</u> equal for $\bar{\omega} \neq 1$.

Power can be understood several ways.

1) As "frictional" power out: $-m\frac{\omega_0}{Q}\langle \dot{x}^2 \rangle$ ... so ...

$$|\bar{P}_{out}| = m\frac{\omega_0}{Q}\,\omega^2 x_0^2\,\frac{1}{2}$$

$$= m\frac{\omega_0}{Q}\,\frac{\omega^2}{(1-\bar{\omega}^2)^2+\left(\frac{\bar{\omega}}{Q}\right)^2}\left(\frac{F_0}{m\omega_0^2}\right)^2\frac{1}{2}$$

... re-arrange ...

$$= \omega_0 Q\,\underbrace{\frac{(\bar{\omega}/Q)^2}{(1-\bar{\omega}^2)^2+\left(\frac{\bar{\omega}}{Q}\right)^2}}_{\sin^2\varphi\,\le\,1}\,\frac{(F_0/\omega_0)^2}{2m}$$

"per time"

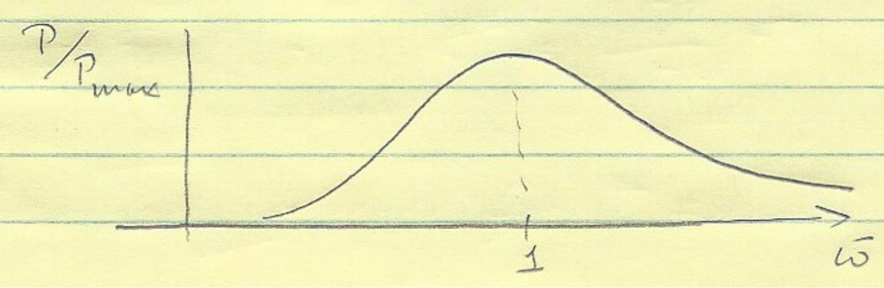
"Lorentz factor"

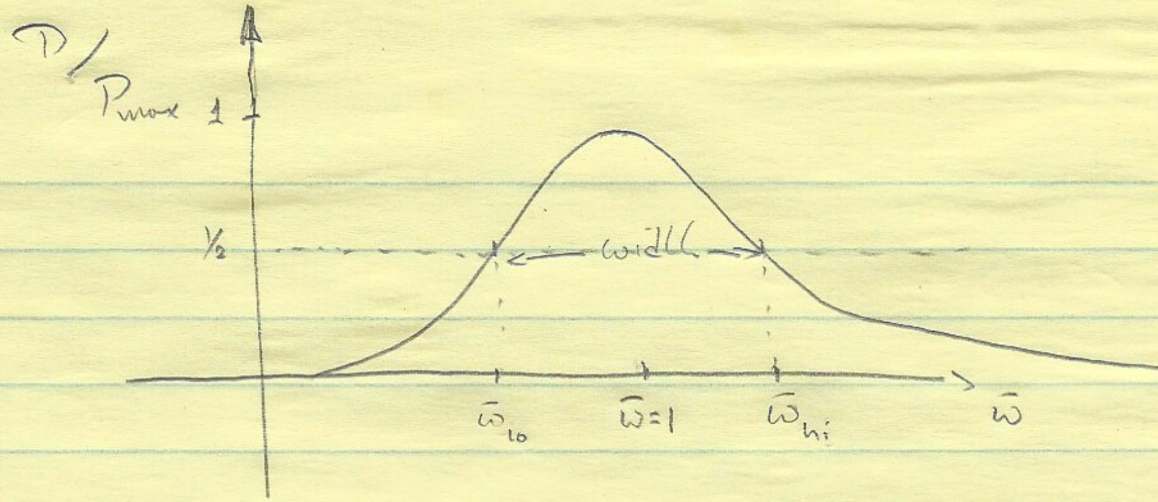
energy

The Lorentz factor is the only "driving frequency" dependency. It is clearly ≤ 1 and has a max value of 1 when $\bar{\omega}=1$.

$$\frac{\bar{P}_{out}(\omega)}{\bar{P}_{out\,max}} = \frac{(\bar{\omega}/Q)^2}{(1-\bar{\omega}^2)^2+(\bar{\omega}/Q)^2}$$

has max of 1 at $\bar{\omega}=1$.
where $\varphi = \pi/2$





$\Delta\bar{\omega}$ = full width at half power happens when

$$(1-\bar{\omega}^2) = \pm\frac{\bar{\omega}}{Q} \qquad \bar{\omega}_{hi} > 1 \;\;\&\;\; \bar{\omega}_{lo} < 1$$

$$\left.\begin{aligned} 1-\bar{\omega}_{hi}^2 &= -\frac{\bar{\omega}_{hi}}{Q} \\ 1-\bar{\omega}_{lo}^2 &= +\frac{\bar{\omega}_{lo}}{Q} \end{aligned}\right\} \text{ subtract!}$$

$$\bar{\omega}_{hi}^2 - \bar{\omega}_{lo}^2 = \frac{\bar{\omega}_{hi}+\bar{\omega}_{lo}}{Q} \quad \ldots \text{ but } a^2-b^2=(a-b)(a+b)\ldots$$

$$\Longrightarrow \bar{\omega}_{hi}-\bar{\omega}_{lo} = \frac{1}{Q}$$

So! $\Delta\bar{\omega} = \dfrac{\Delta\omega}{\omega_0} = \dfrac{1}{Q}$

The resonance curve is centered on $\bar{\omega}$ and has a full width at $\frac{1}{2}$ max of

$$\frac{\Delta\omega}{\omega_0} = \frac{1}{Q}$$

Another way of presenting the power out is:

$$\overline{P}_{out} = -b\langle \dot{x}^2\rangle = -\frac{b}{m}\langle m\dot{x}^2\rangle = -\frac{\omega_0}{Q}2\,\overline{KE}$$

So! $\overline{P}_{out} = -\frac{\omega_0}{Q}\frac{2\bar{\omega}^2}{1+\bar{\omega}^2}\bar{U}$

So $\Delta\bar{U}_{out}$ each cycle is $\overline{P}_{out}T$

$$\Delta U_{cycle} = \overline{P}T = -\frac{2\pi}{Q}\left(\frac{2\bar{\omega}^2}{1+\bar{\omega}^2}\right)\bar{U}$$

or

$$\frac{\Delta U_{cycle}}{\bar{U}} = -\frac{2\pi}{Q}\left(\frac{2\bar{\omega}^2}{1+\bar{\omega}^2}\right)$$

at resonance $\bar{\omega}\to 1$ and, again

$$\frac{\Delta U_{cycle}}{\bar{U}} = -\frac{2\pi}{Q}$$ as in the undriven case!

<u>For the reader</u>: Avg. Power <u>in</u> is $\langle F(t)\,\dot{x}\rangle$

Show that, yes... we get

$$\frac{(F_0/\omega_0)^2}{2m}\,\omega_0 Q\,\sin^2\varphi$$