CSUC
Department of Physics
301A Mechanics: Class Notes

Global Conservation Laws for Systems

I. *INTRODUCTION*

The difficulties that arise in solving even the simplest Newtonian problems are amplified enormously whenever we treat larger collections of particles. This motivates us to seek global theorems describing those relationships which we believe to be of universal validity *irrespective* of the specific complexity at hand. In essence, we believe that, no matter how complicated a system is, certain important features remain simple and we would like to focus in on them at the very outset. The present summary introduces certain of these central results and provides suggestive derivations based on Newton's laws. The standard derivations are, however, open to very many serious objections. Nonetheless, the ensuing discussion has been rich and fruitful and further contributed to the development of analytical mechanics.

II. CENTER OF MASS

The center of mass $\vec{R}_{cm}$ of a collection of point masses $\{m_1, m_2, \ldots, m_n\}$ situated respectively at $\{\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_n\}$ is defined as

$$M_{tot}\,\vec{R}_{cm} \equiv \sum_{i=1}^{n} m_i\,\vec{r}_i, \tag{1}$$

where $M_{tot} \equiv \sum_{i=1}^{n} m_i$. We now conclude, by simple deduction, that

$$\vec{P}_{tot} = \sum_i \vec{p}_i = M_{tot}\,\dot{\vec{R}}_{cm}. \tag{2}$$

Further, we assert that:

$$M_{tot}\ddot{\vec{R}}_{cm} = \dot{\vec{P}}_{tot} = \sum_i \dot{\vec{p}}_i = \sum_{i=1}^{n} \vec{F}_i \;=\; \vec{F}^{net\,external} \tag{3}$$

i.e. the motion of the center of mass is governed by the *net external force contribution* alone. This last result follows from the fact that all forces have identifiable causal origins and may be determined as originating either *interior* or *exterior* to the system. Newton's third law instructs us that the internal forces always occur as members of force-*pairs* that must, then, ultimately cancel pairwise when we sum over *all* forces.

More specifically, that:

$$\sum_{i=1}^{n} \vec{F}_i \;=\; \sum_{i=1}^{n} \left(\vec{F}_i^{\,internal} \;+\; \vec{F}_i^{\,external}\right) \;=\; \vec{F}_i^{\,net\,external} \tag{4}$$

III. ENERGY

We note that, by rearranging equation (1) , we may write

$$\sum_i m_i\,\left(\vec{r}_i - \vec{R}_{cm}\right) = 0 \tag{5}$$

so that it is often convenient to write

$$\vec{r}_i = \vec{R}_{cm} + (\vec{r}_i - \vec{R}_{cm}) = \vec{R}_{cm} + \vec{r}_i{}' \tag{6}$$

where $\vec{r}_i{}'$ is the position vector pointing from the center of mass out to the particle. This vector is called the "relative"position of the particle and this **Center of Mass - Relative** decomposition is of the greatest possible significance and utility. To begin with we note that it is *always* true that:

$$\sum_i m_i \vec{r}_i{}' \equiv 0 \tag{7}$$

so that it will *also* be true that:

$$\frac{d}{dt}\sum_i m_i \vec{r}_i{}' = \frac{d}{dt}(zero) \equiv 0 \tag{8}$$

Notice that this may also be expressed as:

$$\sum_i m_i \dot{\vec{r}}_i{}' \equiv 0 \tag{9}$$

In that case the kinetic energy may be written

$$\begin{aligned} T &= \sum_{i=1}^{n} \frac{m_i}{2} \dot{\vec{r}}_i{}^2 \\ &= \frac{1}{2}\sum_{i=1}^{n} m_i \left(\dot{\vec{R}}_{cm} + \dot{\vec{r}}_i{}'\right) \cdot \left(\dot{\vec{R}}_{cm} + \dot{\vec{r}}_i{}'\right) \\ &= \frac{1}{2} M_{tot}\, \dot{\vec{R}}_{cm} \cdot \dot{\vec{R}}_{cm} + \frac{1}{2}\sum_{i=1}^{n} m_i\, \dot{\vec{v}}_i{}' \cdot \dot{\vec{v}}_i{}' \end{aligned} \tag{10}$$

since, as shown above:

$$\sum_i m_i \dot{\vec{r}}_i{}' = \sum_i m_i \vec{v}_i{}' \equiv 0$$

so that

$$T_{tot} = T_{cm} + T_{rel}$$

where

$$\begin{aligned} T_{cm} &= \frac{1}{2} M_{tot} V_{cm}^2 \\ T_{rel} &= \frac{1}{2}\sum_{i=1}^{n} m_i\, (v_i{}')^2 \end{aligned}$$

and we observe that the **Center of Mass - Relative** decomposition obtains for the *energy* as well.

IV. ANGULAR MOMENTUM

In the same way we may also define the total angular momentum and perform our *CM-Relative* decomposition on it as before:

$$\begin{aligned} \vec{L}_{tot} &\equiv \sum_{i=1}^{n} \vec{r}_i \times \vec{p}_i && (11) \\ &= \sum_{i=1}^{n} (\vec{R}_{cm} + \vec{r}_i{}') \times m_i(\vec{v}_{cm} + \vec{v}_i{}') && (12) \\ &= \vec{R}_{cm} \times M_{tot}\vec{V}_{cm} + \sum_{i=1}^{n} \vec{r}_i{}' \times m_i \vec{v}_i{}' && (13) \\ &= \vec{L}_{cm} + \vec{L}_{rel} && (14) \end{aligned}$$

We observe that just the same process holds here too. Now, taking a time derivative and using the fundamental equation of motion, we see:

$$\begin{aligned}
\frac{d}{dt}\vec{L}_{tot} &= \sum_{i=1}^{n} \vec{r}_i \times \dot{\vec{p}}_i && (15)\\
&= \sum_{i=1}^{n} \vec{r}_i \times \vec{F}_i && (16)\\
&= \sum_{i=1}^{n} (\vec{R}_{cm} + \vec{r}_i{}') \times \vec{F}_i \\
&= \vec{R}_{cm} \times \sum_{i=1}^{n} \vec{F}_i + \sum_{i=1}^{n} \vec{r}_i{}' \times \vec{F}_i && (17)
\end{aligned}$$

By Newton's third law we believe (under the assumption that internal forces cancel pairwise . . .) that:

$$\sum_{i=1}^{n} \vec{F}_i = \vec{F}_i^{net\, external} \tag{18}$$

so that

$$\begin{aligned}
\frac{d}{dt}\vec{L}_{tot} &= \vec{R}_{cm} \times \vec{F}^{net\, ext} + \sum_{i=1}^{n} \vec{r}_i{}' \times \vec{F}_i \\
\frac{d}{dt}(\vec{L}_{cm} + \vec{L}_{rel}) &= \vec{R}_{cm} \times \vec{F}^{net\, ext} + \sum_{i=1}^{n} \vec{r}_i{}' \times \vec{F}_i && (19)
\end{aligned}$$

But since independently (see above) we have:

$$\frac{d}{dt}\vec{L}_{cm} = \vec{R}_{cm} \times \vec{F}^{net\, ext} \tag{20}$$

we can combine this with equation (19) to achieve the further result:

$$\frac{d}{dt}\vec{L}_{rel} = \sum_{i=1}^{n} \vec{r}_i{}' \times \vec{F}_i = \sum_{i=1}^{n} \vec{r}_i{}' \times \left(\vec{F}_i^{internal} + \vec{F}_i^{external}\right) \quad . \tag{21}$$

Here we have, once again, made the decomposition of each force $\vec{F}_i = \left(\vec{F}_i^{internal} + \vec{F}_i^{external}\right)$ into its contributions from inside or outside the system. Finally, then, equation (21) is now expressed as

$$\frac{d}{dt}\vec{L}_{rel} = \sum_{i=1}^{n} \vec{r}_i{}' \times \vec{F}_i^{ext} \tag{22}$$

under the argument that the net-sum of all ***internally*** generated torques *must also* total up to zero.

V. SUMMARY

By way of summary, we collect below the most important conclusions from this discussion.

$$
\begin{aligned}
M_{tot} &\equiv \sum_{i=1}^{n} m_i \\
M_{tot}\,\vec{R}_{cm} &\equiv \sum_{i=1}^{n} m_i\,\vec{r}_i \\
\vec{r}_i &= \vec{R}_{cm} + \vec{r}_i{}'
\end{aligned}
$$

$$
\begin{aligned}
\vec{P}_{tot} &= \sum_i \vec{p}_i = M_{tot}\,\dot{\vec{R}}_{cm} \\
M_{tot}\ddot{\vec{R}}_{cm} &= \dot{\vec{P}}_{tot} = \vec{F}^{net\,external}
\end{aligned}
$$

$$
\begin{aligned}
T_{tot} &= T_{cm} + T_{rel} \\
T_{cm} &= \frac{1}{2} M_{tot} V_{cm}^2 \\
T_{rel} &= \frac{1}{2} \sum_{i=1}^{n} m_i\,(v_i{}')^2
\end{aligned}
$$

$$
\begin{aligned}
\vec{L}_{tot} &\equiv \sum_{i=1}^{n} \vec{r}_i \times \vec{p}_i \\
&= \vec{R}_{cm} \times M_{tot}\vec{V}_{cm} + \sum_{i=1}^{n} \vec{r}_i{}' \times m_i \vec{v}_i{}' \\
&= \vec{L}_{cm} + \vec{L}_{rel}
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{dt}\vec{L}_{cm} &= \vec{R}_{cm} \times \vec{F}^{net\,ext} \\
\frac{d}{dt}\vec{L}_{rel} &= \sum_{i=1}^{n} \vec{r}_i{}' \times \vec{F}_i^{ext}
\end{aligned}
$$