Taylor Series ... or "An expansion in Small"

Overview:

The concept of a Taylor's Series for a function $f(x)$ is based on the observation that, if only a function is sufficiently smooth, the function may be approximated by a polynomial (plus remainder) in a limited interval about some starting point. The coefficients for the various powers in the polynomial are expressed in terms of derivatives of the function evaluated at the base point. The number of terms in the polynomial is limited only by the number of derivatives that exist. Indeed, if derivatives to all orders exist, we may express an infinite series that represents the function in some interval of convergence. If this series exists, it is unique.

In general we write:

$$f(x+a) = f(x) + \frac{a^1}{1!} f'(x) + \frac{a^2}{2!} f'' + \cdots + \frac{a^n}{n!} f^{[n]} + R$$

Here, the base point is "x", ... the "excursion away from x" is "a"; "n" is the highest derivative, and R is the remainder.

The various proofs will be given later, but the intent is clear. As ever more terms are included we achieve an ever better approximation to our function. If derivatives to all orders exist, then we have found the (unique) infinite series representing our function.

<u>Trick:</u>

If we write $f'(x)$ as $\frac{d}{dx} f$ and $f''(x)$ as $\frac{d^2}{dx^2} f$ etc.

it is inviting to write:

$$f(x+a) = \left\{ 1 + \frac{1}{1!} a \frac{d}{dx} + \frac{1}{2!} \left(a \frac{d}{dx}\right)^2 + \cdots \frac{1}{n!} \left(a \frac{d}{dx}\right)^n \right\} f + R$$

The form of the series is <u>very suggestive</u>, and if continued indefinitely can be remembered as:

$$f(x+a) = e^{a \frac{d}{dx}} f(x)$$

where the "exponential" is understood as a formal power series

$$e^x \equiv 1 + \frac{1}{1!} x + \frac{1}{2!} x^2 + \cdots .$$

<u>Multivariable Taylor Expansion.</u>

Consider a function of a vector argument, e.g. $f(\vec{v})$. Is it possible to extend our formalism here? The answer is <u>certainly</u>, nor is it difficult. Consider making a new function $g(s) \equiv f(\vec{v} + s\vec{a})$. Then $g(0) = f(\vec{v})$ and $g(1) = f(\vec{v} + \vec{a})$. But $g(s) = g(x+s)\Big|^{x=0}$

and $g(x+s) = e^{s \frac{d}{dx}} g(x)$.

So, we wish to find $f(\vec{v}+\vec{a}) = e^{s \frac{d}{dt}} g(t)\Big|^{t=0}\Big|^{s=1}$.

Observe! $\frac{d}{dt} g(t) = \frac{d}{dt} f(\vec{v}_0 + t\vec{a})$.

Think of $\vec{v}_0 + t\vec{a}$ as $\vec{v}(t)$ where $\frac{d\vec{v}(t)}{dt} = \vec{a}$

so $\frac{d}{dt} f(\vec{v}(t)) = \frac{dx}{dt}\frac{\partial f}{\partial x(t)} + \frac{dy}{dt}\frac{\partial f}{\partial y(t)} + \frac{dz}{dt}\frac{\partial f}{\partial z(t)}$

$$= \frac{d\vec{v}(t)}{dt} \cdot \vec{\nabla} f(\vec{v}(t)) = \vec{a} \cdot \vec{\nabla} f$$

since $\vec{v}(t=0) = \vec{v}_0$ (our base or "starting" point).

Then $\frac{d}{dt} f(\vec{v}(t)) \Big|^{t=0} = \vec{a} \cdot \vec{\nabla} f \Big|^{\vec{v}_0}$

and likewise $\left(\frac{d}{dt}\right)^n f(\vec{v}(t)) \Big|^{t=0} = (\vec{a} \cdot \vec{\nabla})^n f(\vec{v}) \Big|^{\vec{v}_0}$

and $f(\vec{v} + s\vec{a}) = e^{s\vec{a} \cdot \vec{\nabla}} f(\vec{v}) \Big|^{\vec{v}_0}$

Now set $s = 1$ and we are done!

$f(\vec{v} + \vec{a}) = e^{\vec{a} \cdot \vec{\nabla}} f(\vec{v})$. Observe that

$$a\frac{d}{dx} \rightsquigarrow a_x \frac{\partial}{\partial x} + a_y \frac{\partial}{\partial y} + a_z \frac{\partial}{\partial z} = \vec{a} \cdot \vec{\nabla}$$

is the natural transition from 1-D to 3-D etc.

Practice with gradients!

Let $\vec{c}$ be a constant vector and $r \equiv |\vec{r}|$,
Prove the following!

✓ $\vec{\nabla}(\vec{c}\cdot\vec{r}) = \vec{c}$

✓ $\vec{\nabla} r = \frac{\vec{r}}{r} \;(= \hat{r})$

✓ $\vec{\nabla} r^n = n r^{n-1} \hat{r}$

✓ $\vec{\nabla} fg = \vec{\nabla} f \cdot g + f \vec{\nabla} g$

✓ $\vec{\nabla}\, \vec{c}\cdot\hat{r} = \frac{\vec{c}}{r} - \frac{\vec{c}\cdot\vec{r}}{r^3}\vec{r}$

Practice with Taylor Expansions.

$$|\vec{r}+\vec{s}| = ? = |\vec{r}| + \frac{(\vec{s}\cdot\vec{\nabla})^1}{1!}|\vec{r}| + \frac{(\vec{s}\cdot\vec{\nabla})^2}{2!}|\vec{r}| + \text{h.p.t.}$$

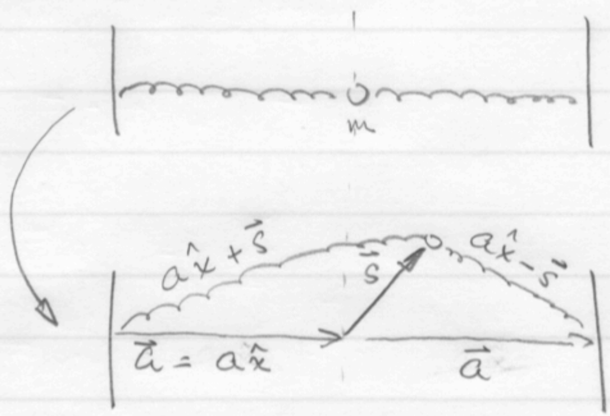
use $\vec{s}\cdot\vec{\nabla}|\vec{r}| = \vec{s}\cdot\hat{r}$ ... so ...

$$(\vec{s}\cdot\vec{\nabla})^2|\vec{r}| = \vec{s}\cdot\vec{\nabla}\;\vec{s}\cdot\hat{r} = \vec{s}\cdot\left(\frac{\vec{s}}{r} - \frac{\vec{s}\cdot\vec{r}}{r^3}\vec{r}\right)$$

$$= \frac{\vec{s}^{\,2}}{r} - \frac{(\vec{s}\cdot\vec{r})^2}{r^3}$$

$$\text{So } |\vec{r}+\vec{s}| = r + \underbrace{\frac{1}{1!}\frac{\vec{s}\cdot\vec{r}}{r}}_{1^{st}\text{ order in }\vec{s}} + \underbrace{\frac{1}{2!}\left(\frac{\vec{s}^{\,2}}{r} - \frac{(\vec{s}\cdot\vec{r})^2}{r^3}\right)}_{2^{nd}\text{ order in }\vec{s}} + \text{h.p.t.}$$

Hints for 5.18



Starting positions for our two springs.

Now we displace "m" by $\vec{s}$.

the left hand spring has length: $|\vec{a}+\vec{s}| \equiv \ell_1$
the right hand spring has length $|\vec{a}-\vec{s}| \equiv \ell_2$

Each spring has a *rest* length of ℓ_0.

The potential energy in the springs is

$$\frac{1}{2}k(\ell_1-\ell_0)^2 + \frac{1}{2}k(\ell_2-\ell_0)^2$$

Expand these expressions to 2nd order in $\vec{s}$ ...
... and watch if the energy goes up or down!