The Taylor Expansion

A nice "old fashioned" demonstration of the Taylor Expansion uses differentials and their manipulations which are, in any case, good practice to learn.

The central insight is merely to recognize that integration by parts is just the "product-rule" of differentials. We notice, then:

$d(fg) = df\,g + f\,dg$ so that, rearranging

$d(fg) - df\,g = f\,dg$ so, now...

combine this with $\int_{x=a}^{x=b} d(F(x)) = F(x)\Big|_{x=a}^{x=b}$

and we achieve $\int_{x=a}^{x=b} f(x)\,d(g(x)) = f(x)\,g(x)\Big|_{x=a}^{x=b} - \int_{a}^{b} df(x)\,g(x)$

The trick is, then, to "integrate by parts" repeatedly and pull off the total differential terms. This builds a power series which is the Taylor expansion.

We start by listing the basic rules for manipulating differentials.

I. The Rules:

1) $d(f(x)) = f'(x)\,dx$

2) $dx = d(x-b)$ for any constant b.

3) $f\,dg = d(fg) - df\,g$

3') $f(x)\,dx = f(x)\,d(x-b) = d\Big(f(x)(x-b)\Big) - df(x)\,(x-b)$

4) $(x-b)^n\,dx = (x-b)^n\,d(x-b) = d\,\dfrac{(x-b)^{n+1}}{n+1}$

5) $\displaystyle\int_{x=a}^{x=b} d(f(x)) = f(b) - f(a)$

II. The Procedure: We apply the rules again, & again.

$$df(x) = f'(x)\,dx = f'(x)\,d(x-b)$$

$$= d\Big(f'(x)(x-b)\Big) - \underbrace{d\big(f'(x)\big)}(x-b)$$

$$= \quad '' \quad - f''(x)\,dx\,(x-b)$$

$$= \quad '' \quad - f''(x)\,d\,\frac{(x-b)^2}{2}$$

, we achieve

$$df(x) = d\Big(f'(x)(x-b)\Big) - f''(x)\,d\,\frac{(x-b)^2}{2}$$

Notice especially that we have employed rule 3) to move the differential symbol "d" off its starting position and over to the other (multiplying) function. This creates a new "total differential" and a left over term which becomes the new focus.

III We do it again!

$$f''(x)\, d\left(\frac{(x-5)^2}{2}\right) = d\left(f''(x)\,\frac{(x-5)^2}{2}\right) - df''(x)\,\frac{(x-5)^2}{2}$$

But $df''(x) = f'''(x)\,dx$ and $dx\,\frac{(x-5)^2}{2} = d\left(\frac{(x-5)^3}{3!}\right)$

$$\text{So } f''(x)\, d\left(\frac{(x-5)^2}{2!}\right) = d\left(f''(x)\,\frac{(x-5)^2}{2!}\right) - f'''(x)\, d\left(\frac{(x-5)^3}{3!}\right)$$

Now we have, sum total:

$$df(x) = d\left(f'(x)\,(x-5)\right) - d\left(f''(x)\,\frac{(x-5)}{2!}\right) + f'''(x)\, d\left(\frac{(x-5)^3}{3!}\right)$$

IV We take the last term and do it <u>again</u>... (and again)

$$df(x) = d\left\{\sum_{k=1}^{n} (-1)^{k-1} f^{(k)}(x)\,\frac{(x-5)^k}{k!}\right\} + (-1)^n f^{(n+1)}(x)\, d\,\frac{(x-5)^{n+1}}{(n+1)!}$$

V. Conclusion.

Now we use rule 6) which says that the integral of a differential of an argument is just the argument itself evaluated across the endpoints. That is, we integrate the statement in IV. from $a \to b$. $\int_a^b [\quad]$. We achieve,

$$f(b) - f(a) = \sum_{k=1}^{n} f^{(k)}(a) \frac{(b-a)^k}{k!} + (-1)^n \int_a^b f^{(n+1)}(x) \frac{d(x-b)^{n+1}}{(n+1)!}$$

If we call the upper limit 'x' we find:

$$f(x) = \sum_{k=0}^{n} \frac{(x-a)^k}{k!} f^{(k)}(a) + (-1)^n \int_a^x f^{(n+1)}(x') \frac{d(x'-x)^{n+1}}{(n+1)!}$$

It is this form which is so commonly found.

If our remainder term is "small", we can use the first 'n' power terms as a polynomial approximation of our function $f(x)$