Taylor Series ... and Variational ... Notation!

Our starting point in the discussion is Taylor's expansion:

i) $f(x+a) = f(x) + \frac{1}{1!} a^1 \frac{d}{dx} f(x) + \frac{1}{2!} a^2 \frac{d^2}{dx^2} f(x) + \cdots$

ii) $f(\vec{v}+\vec{a}) = f(\vec{v}) + \frac{1}{1!} \vec{a}\cdot\vec{\nabla} f(\vec{v}) + \frac{1}{2!} (\vec{a}\cdot\vec{\nabla})^2 f(\vec{v}) + \cdots$

We are supposed to notice that the increment 'a' or '$\vec{a}$' only appears in combination with a derivative. That is, we only see

$$a\frac{d}{dx} \quad \text{or} \quad \vec{a}\cdot\vec{\nabla}.$$

We may think of this combination as "dimensionless". It also reminds us of taking "differentials".

Recall, $df(x) = dx \frac{df}{dx} \longleftrightarrow dx \frac{d}{dx} f(x)$.

So, $a\frac{df}{dx}$ is very like $dx\frac{df}{dx}$ only our increment "a" is finite and not infinitesmal.

At this point we introduce a new notation: just as $df(x) = dx \frac{d}{dx} f(x)$, we now write

$$\delta f(x) = \delta x \frac{d}{dx} f(x)$$

δx plays the role of the increment 'a', and is supposed to remind us of the differential "advance", only now it is <u>finite</u>.

Now, also, we will find ourselves taking "first" variations and "second" variations and even higher ones.

$$\text{eg.} \quad \delta f(x) = \delta x \frac{df}{dx} = \left(\delta x \frac{d}{dx}\right)^1 f$$

$$\delta^2 f(x) = \left(\delta x \frac{d}{dx}\right)^2 f$$

$$\vdots$$

$$\delta^n f(x) = \left(\delta x \frac{d}{dx}\right)^n f$$

Our Taylor series now has a lovely form:

$$f(x + \delta x) = f(x) + \frac{1}{1!} \delta f + \frac{1}{2!} \delta^2 f + \cdots .$$

What We have gained:

1) taking differentials expresses the inner truth about change, without explicit reference to derivatives. This is what variations do.

2) Differentials & Variations are dimensionless!

3) The 'nth' variation finds the (unique) contribution having 'n' factors of the increment. The uniqueness stems from the uniqueness of power series.

4) Single variable & multivariable expressions are now unified!

5) Our logic works forward & backward!

Since I know $\delta f(\vec{v}) = \delta\vec{v} \cdot \vec{\nabla} f$ ...

Then, if I find that $\delta f = \delta\vec{v} \cdot \vec{A}$, it follows that $\vec{\nabla} f \equiv \vec{A}$! We will greatly simplify the finding of derivatives!

Practice Examples.

1) Suppose $f(\vec{v}) = |\vec{v}| = \sqrt{x^2+y^2+z^2} = \sqrt{\vec{v}\cdot\vec{v}}$

What is $f(\vec{v}+\vec{a})$? $= f(\vec{v}) + \frac{1}{1!}\delta f + \frac{1}{2!}\delta^2 f + \cdots$

Use a trick! $f^2 = \vec{v}\cdot\vec{v}$ and "taking variations" is identical to "taking differentials".

start: $f^2 = \vec{v}\cdot\vec{v} \Rightarrow 2f\,\delta f = 2\vec{v}\cdot\delta\vec{v}$

so $f\,\delta f = \vec{v}\cdot\delta\vec{v}$ (now take 2nd variation)

$\longrightarrow (\delta f)^2 + f\,\delta^2 f = \delta\vec{v}\cdot\delta\vec{v}$ etc

We recover ✓ $f(\vec{v}) = |\vec{v}|$

✓ $\delta f = \dfrac{\vec{v}\cdot\delta\vec{v}}{|\vec{v}|} = \hat{v}\cdot\delta\vec{v}$

✓ $\delta^2 f = \dfrac{\delta\vec{v}\cdot\delta\vec{v} - (\delta f)^2}{f}$

$\hookrightarrow \delta^2 f = \dfrac{|\delta\vec{v}|^2 - (\hat{v}\cdot\delta\vec{v})^2}{v}$

So $f(\vec{v}+\delta\vec{v}) = v + \frac{1}{1!}\hat{v}\cdot\delta\vec{v} + \frac{1}{2!}\left(\dfrac{|\delta\vec{v}|^2 - (\hat{v}\cdot\delta\vec{v})^2}{v}\right) + \cdots$

our expansion!