Week 11 Notes: Momentum.

Intro:

Energy and Momentum are at the very core of classical mechanics. Both are conserved. We use the letter $\vec{p}$ exclusively for momentum. Newton used the expession $m\vec{v}$ as his representation of the linear momentum of a mass m: $\vec{p} = m\vec{v}$.

The second law now becomes fully:

$$\frac{d\vec{p}}{dt} = \vec{F}^{net}$$

I would like to accentuate 3 categories of problems, perhaps moreso than our text.

1) Center of Mass.

2) Collisions

3) Variable mass problems.

Center of Mass.

The Center of Mass of a collection of particles is a location ... the average position $\vec{R}_{cm}$

Suppose we have a system of particles:
$\{m_1, m_2, m_3, \dots m_N\}$

each at its own position $\{\vec{r}_1, \vec{r}_2, \dots \vec{r}_N\}$

We define $M_{tot} \equiv \sum_{i=1}^{N} m_i$

$$M_{tot} \vec{R}_{cm} \equiv \sum_{i=1}^{N} m_i \vec{r}_i$$

Some immediate consequences follow.

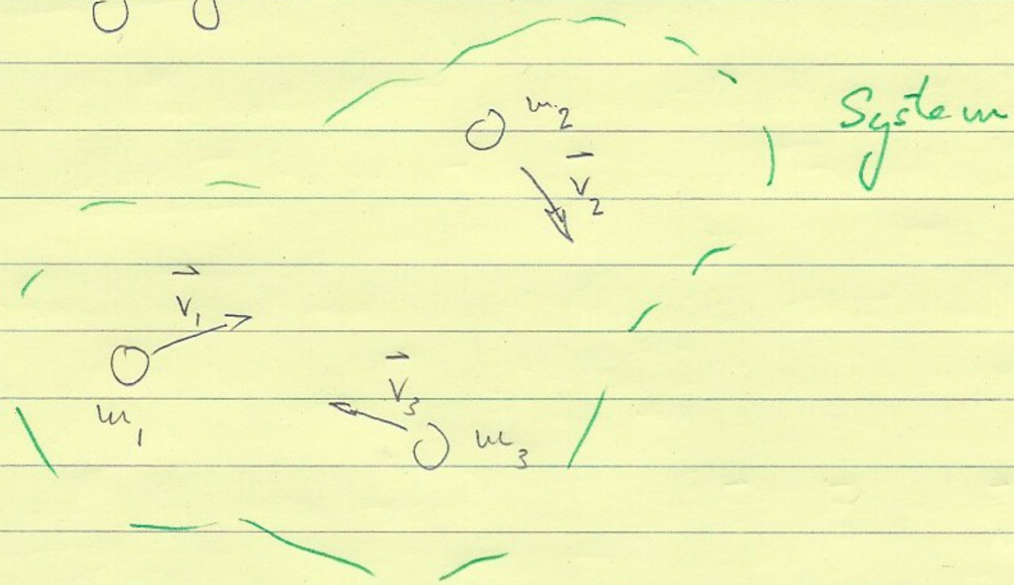
$$\frac{d}{dt}(M_{tot} \vec{R}_{cm}) = M_{tot} \vec{V}_{cm} = \sum_{i=1}^{N} m_i \vec{v}_i = \sum_{i=1}^{N} \vec{P}_i$$

So! $M_{tot} \vec{V}_{cm} = \vec{P}_{tot} \quad \left(= \sum_{i=1}^{N} \vec{P}_i \right)$

So also $\frac{d}{dt}(M_{tot} \vec{V}_{cm}) = M_{tot} \vec{a}_{cm} = \frac{d}{dt} \sum_{i=1}^{N} \vec{P}_i$

but ... $\frac{d}{dt} \vec{P}_i = \vec{F}^{net}$

$\vec{F}$or any system ...



... forces between particles come in pairs:

and, e.g. $\vec{F}_{12} = -\vec{F}_{21}$

So when we add up all the force on our system of particles ... all the internal forces cancel in pairs.

$$\vec{F}^{net} = \sum_{i=1}^{N} \vec{F}_i^{Net} = \sum_{i=1}^{N} \vec{F}_i^{net\,ext} = \vec{F}^{net\,ext}$$

We conclude that:

$$\frac{d\vec{p}^{\,tot}}{dt} = M_{tot} \frac{d}{dt} \vec{V}_{cm} = \vec{F}^{net\,ext}.$$

Only external forces can change a system's net momentum ... and/or accelerate its Center of Mass.

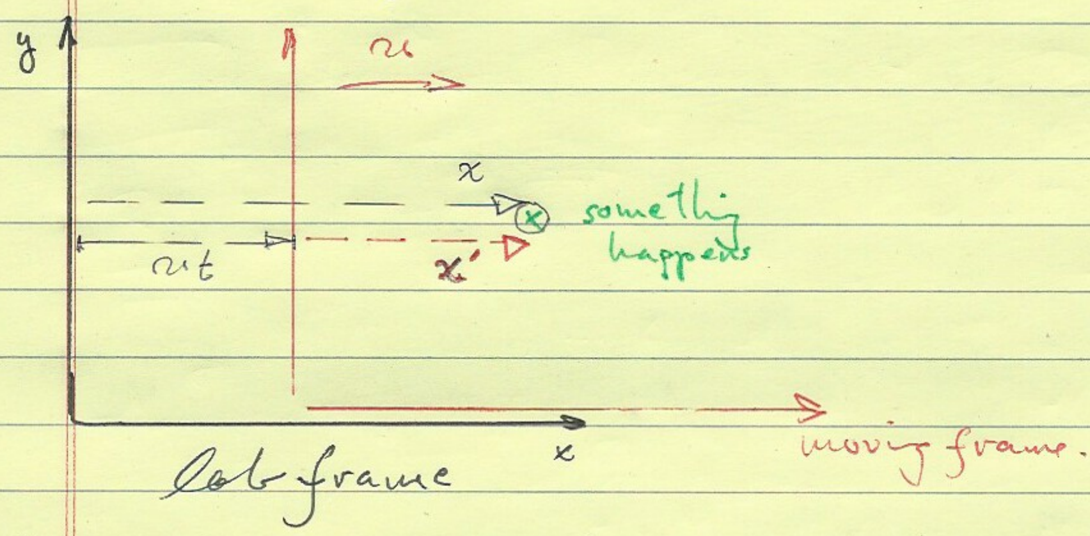
The Center of Mass Reference Frame.

The Laws of Physics are true. But there's more... They are the same in every inertial reference frame. Remember - physics is about relationships. In different frames of reference ... the numbers may be different ... but the relationships between the numbers in any one frame is the same relationship for every reference frame. No matter what the numbers turn out to be ... Energy, & momentum are conserved the same way in every frame. This is knowledge of real power. If we have a system interacting with itself ... but isolated from the outside ... then ... the total momentum can't change. But! $\vec{P}_{tot} = M_{tot} \vec{V}_{cm}$.

So we conclude that the center of mass velocity is constant. This being so ... we can attach a frame of reference to it. In this special frame ... the value of the center of mass velocity is zero! Accordingly, in this frame the total momentum is zero too. Zero is a special number! Many arithmetic results become so much simpler in this frame.

We will use this knowledge to solve the "totally elastic collision" problem.

Changing Reference frames.



When something happens ... we locate it with a coordinate number. Suppose we have two frames, the red one moving to the right at speed u relative to the black frame. They will locate the "green event" shown with different numbers.

The black "lab frame" will use x
The red "moving frame" will use x'

We see that $x = x' + ut$.
If the "green event" is moving as time passes we can locate it twice

$$\left.\begin{aligned} x_1 &= x_1' + ut_1 \\ x_2 &= x_2' + ut_2 \end{aligned}\right\} \text{subtract}$$

$$\Delta x = \Delta x' + u\,\Delta t \quad \rightsquigarrow \quad \frac{\Delta x}{\Delta t} = \frac{\Delta x'}{\Delta t} + u$$

or $v' = v - u$; same events, different measured velocities.

Collisions

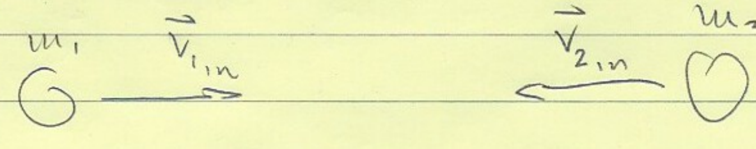
In collisions between particles both momentum and energy are conserved. The problem with energy is that it can be stored up many ways. Momentum will have only one form. We distinguish a couple of limiting cases:

a) Totally inelastic: This is where the particles join.

b) Totally Elastic: This is where all the incoming kinetic energy stays kinetic.

Our central task is to use conservation of momentum & energy to predict how particles emerge from a collision ... assuming we know how they went in!

e.g.



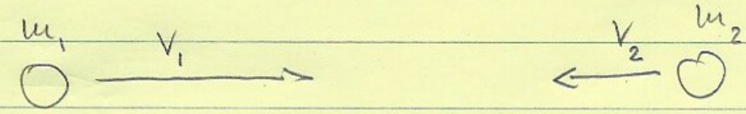
Task: find $\vec{v}_{1_{out}}$ and $\vec{v}_{2_{out}}$ ...

if we know m_1, m_2, $\vec{v}_{1_{in}}$ & $\vec{v}_{2_{in}}$.

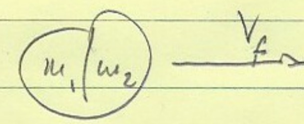
Totally Inelastic Collisions

Particles meet & join!

Before:



After



but ... momentum is conserved!

$$\Longrightarrow \quad m_1 v_1 + m_2 v_2 = (m_1 + m_2)\, v_f = P^{tot} = M_{tot}\, V_{CM}$$

So v_f $\underline{\underline{is}}$ the center of mass velocity.

If we move the right hand side to the left ...

$$m_1 (v_1 - v_f) + m_2 (v_2 - v_f) = 0$$

but $v_f = v_{CM}$ and $\left.\begin{matrix} v_1 - v_{CM} \\ v_2 - v_{CM} \end{matrix}\right\}$ $\underline{\underline{are}}$ the velocities we would use in the C.M. frame.

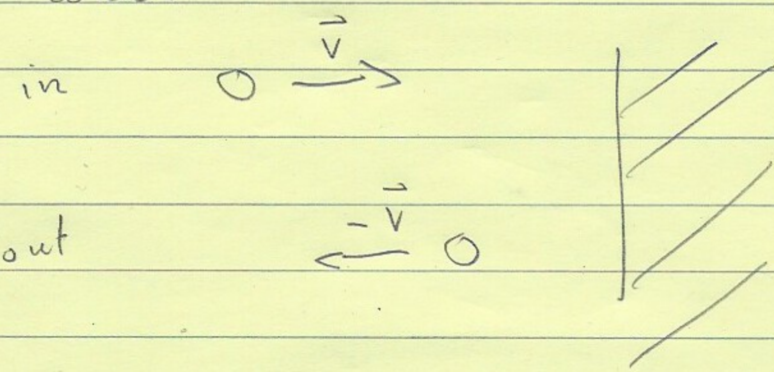
In that frame ... the particles are seen as coming to rest. Also, we observe ... in the C.M. frame all the incoming Kinetic Energy disappears ... and must have become internal energy (maybe heat etc.)

How much K.E went inside?

$$\tfrac{1}{2} m_1 (v_1 - v_{CM})^2 + \tfrac{1}{2} m_2 (v_2 - v_{CM})^2 \Longrightarrow \text{Internal Energy.}$$

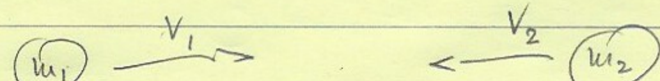
Elastic Collisions

The simplest ~~single~~ single particle elastic collision is just a "perfect superball" bouncing off a hard wall:

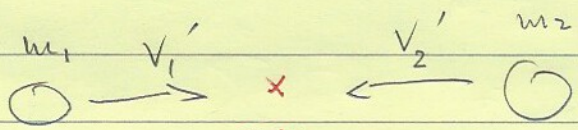


Kinetic energy has been held constant.

How about two particles?



what does this look like in C.M. frame?



is at rest!

Since $P_1' + P_2' = 0$ (C.M. is the zero net momentum frame)

we also have $P_{1_f}' + P_{2_f}' = 0$ or $P_{2_f}' = -P_{1_f}'$

but $(KE)_{\text{initial}} = (KE)_{\text{final}}$ for elastic collisions.

So $$\frac{P_1'^2}{2m_1} + \frac{P_2'^2}{2m_2} = \frac{P_{1_f}'^2}{2m_1} + \frac{P_{2_f}'^2}{2m_2}$$

but! $P_1'^2 = P_2'^2$ & $P_{1_f}'^2 = P_{2_f}'^2$

So! $$P_1'^2\left(\frac{1}{2m_1} + \frac{1}{2m_2}\right) = P_{1_f}'^2\left(\frac{1}{2m_1} + \frac{1}{2m_2}\right)$$

So $P_{1_f}' = \pm P_1'$

+ sign? It went through! (Can this happen?)

− sign? It bounced back like a super ball.

We take the minus sign as our case.

The solution to the problem is simple in this frame.

$$V_{1_f}' = -V_1'$$

$$V_{2_f}' = -V_2'$$

or, in terms of the lab velocities:

$$\left.\begin{aligned}(V_{1_f} - V_{cm}) &= -(V_1 - V_{cm})\\(V_{2_f} - V_{cm}) &= -(V_2 - V_{cm})\end{aligned}\right\} \longrightarrow \begin{aligned}V_{1_f} &= -V_1 + 2V_{cm}\\V_{2_f} &= -V_2 + 2V_{cm}\end{aligned}$$

Our Answer!

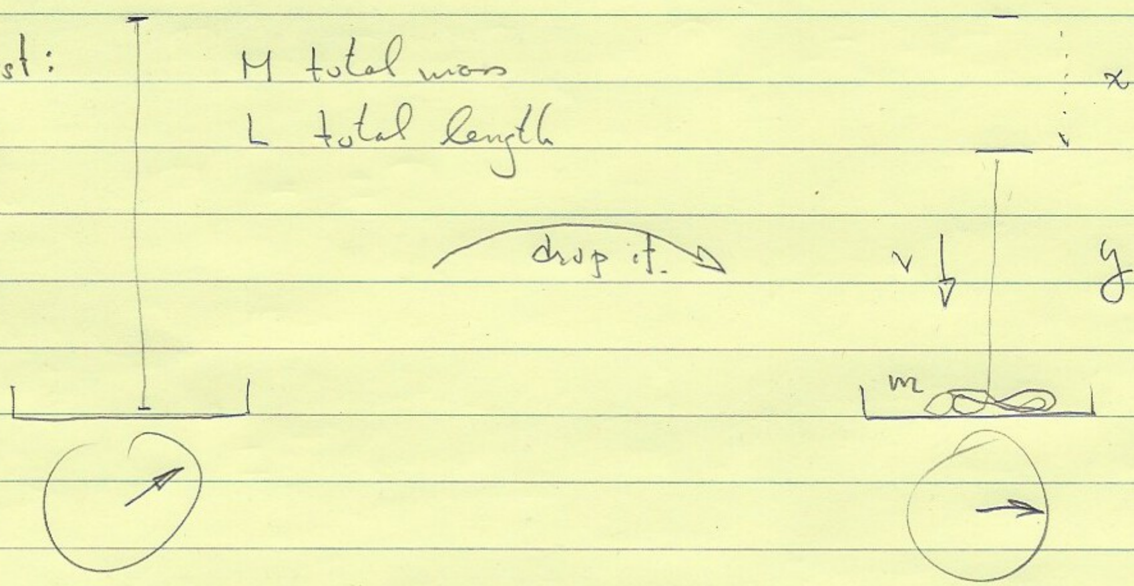
Variable Mass

✓ Moving mass has momentum.

✓ The transfer of momentum over time *is* force.

As a classic problem ... consider a falling chain

Start at rest:



When the chain has fallen a distance x ...

✓ $m = \frac{x}{L} M$ is in the scale pan.

✓ The remaining column of height y has $v^2 = 2gx$

In the next dt of time, $dm = \frac{dx}{L} M = \frac{v\,dt}{L} M$
enters the pan

It brings in momentum $dm\,v = \frac{v^2}{L} M\,dt = \frac{2gx}{L} M\,dt$

The mass *already* in the pan brings in $dp = mg\,dt$

total $dp = mg\,dt + \frac{2gx}{L} M\,dt = 3mg\,dt$

$\frac{dp}{dt} = 3mg$! The scale reads 3x weight!