Notes for Week 12 : Rotational Motion.

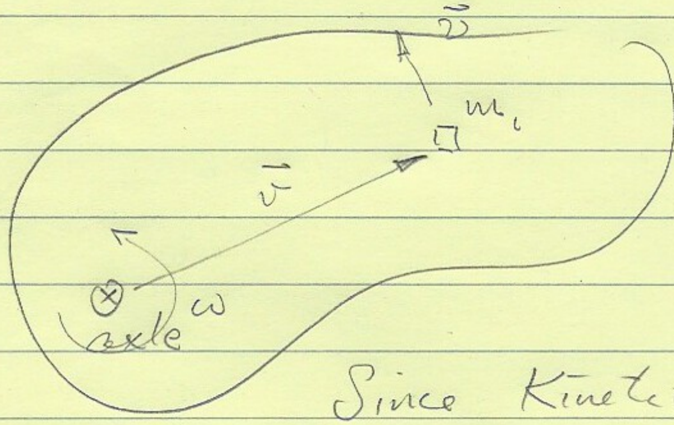
This week we have two central ideas to connect:

1) Moment of Inertia (and the parallel axis thm.)

2) Torque ... which brings an inertia into motion. We use the W.E. theorem to establish the relationship

Both of those start from Kinetic energy!

For a pt. mass $K.E. = \frac{1}{2} m \vec{v}^2$ ... how about a rigid body? ... made up of many masses m_i.



Suppose we have a rigid object rotating about an axle at rate ω :

Since Kinetic Energy is additive ...

$$KE_{tot} = \sum_{\substack{\text{masses} \\ i}} \frac{1}{2} m_i v_i^2 \quad \ldots \text{ but } |\vec{v}_i| = \omega |\vec{r}_i|$$

(we write $v_i = \omega r_i$)

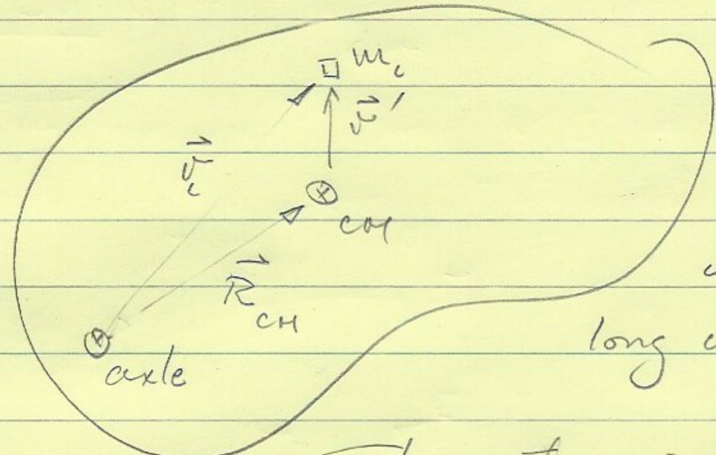
$$KE_{tot} = \frac{1}{2} \sum_i m_i (r_i \omega)^2 = \frac{1}{2}\left(\sum_i m_i r_i^2\right) \omega^2$$

We define $I \equiv \sum_i m_i r_i^2$ = "moment of Inertia"

Moment of inertia I will represent for rotation what mass does for translation:
... "resistance to change of motion".

Computing I will be a persistent task ... and also one full of tricks! Later in these notes we will explore some of these tricks.

1st a central result: the parallel axis thm.



Consider two choices for an axle ... the C.M. and anywhere else ... as long as the axles are parallel.

The vector pointing to the C.M. location from any chosen origin is given by

$$M_{tot}\,\vec R_{CM} \equiv \sum_i m_i \vec r_i$$

Since $M_{tot} = \sum_i m_i$ ... $\Longrightarrow 0 = \sum_i m_i(\vec r_i - \vec R_{CM})$

Let $\vec r_i - \vec R_{CM} \equiv \vec r_i\,'$ so $\vec r_i = \vec R_{CM} + \vec r_i\,'$

Then $\sum_i m_i \vec r_i\,' \equiv 0$.

Now insert $\vec r_i = \vec R_{CM} + \vec r_i\,'$ into I definition.

$$I = \sum_i m_i r_i^2 = \sum_i m_i \vec{r}_i \cdot \vec{r}_i = \sum_i m_i (\vec{r}_i{}' + \vec{R}_{cm}) \cdot (\vec{r}\,' + \vec{R}_{cm})$$

$$= \sum_i m_i \left\{ \vec{r}_i{}' \cdot \vec{r}_i{}' + 2\vec{r}_i{}' \cdot \vec{R}_{cm} + \vec{R}_{cm} \cdot \vec{R}_{cm} \right\}$$

but $\sum_i m_i 2\vec{r}_i{}' \cdot \vec{R}_{cm} = 2\vec{R}_{cm} \cdot \sum_i m_i \vec{r}_i{}' = 0$!

$\vec{r}_i{}' \cdot \vec{r}_i{}' = r_i^2$; $\vec{R}_{cm} \cdot \vec{R}_{cm} = R_{cm}^2$

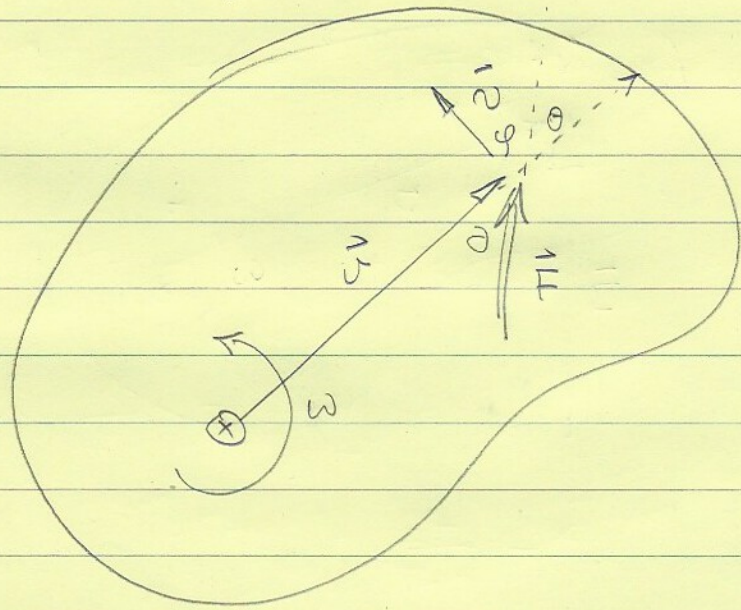
and $\sum_i m_i r_i^2 \equiv I_{cm}$; $\sum m_i R_{cm}^2 = M_{tot} R_{cm}^2$

so! $I = I_{cm} + M_{tot} R_{cm}^2$

Meaning: The moment of inertia about any axis is equal to the moment of inertia about a parallel axis through the Center of Mass plus a correction term $M_{tot} R_{cm}^2$. This correction is the moment of inertia of a point mass at the C.M. location of value M_{tot}.

Think this way: "all objects act like a point mass M_{tot} at the C.M. and an object of inertia I_{cm} rotating about the C.M."

The equations of rotational motion.



The equation of motion is the work energy theorem.

$$dW = d\,(K.E.)$$

or $$\frac{dW}{dt} = \frac{d}{dt}\left(\frac{1}{2} I \omega^2\right)$$

$$\frac{dW}{dt} = \vec{F}\cdot\vec{v} = I\,\omega\,\frac{d\omega}{dt}$$

but $\vec{F}\cdot\vec{v} = |\vec{F}||\vec{v}|\cos\varphi$ and $|\vec{v}| = \omega|\vec{r}|$

and $\cos\varphi = \sin\theta$

so! $F\,\omega\, r \sin\theta = I\,\omega\,\frac{d\omega}{dt}$ (cancel ω)

$$F\, r \sin\theta = I\,\alpha$$

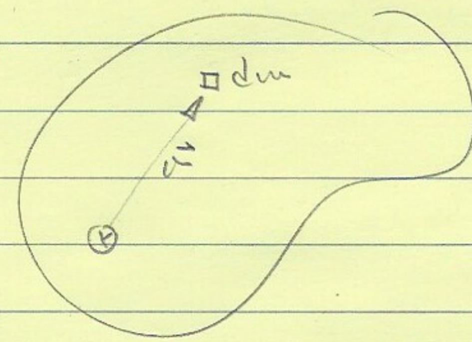
since $F\sin\theta = F_\perp$ & $r\sin\theta = r_\perp$

torque $= F r \sin\theta = F r_\perp = F_\perp r$.

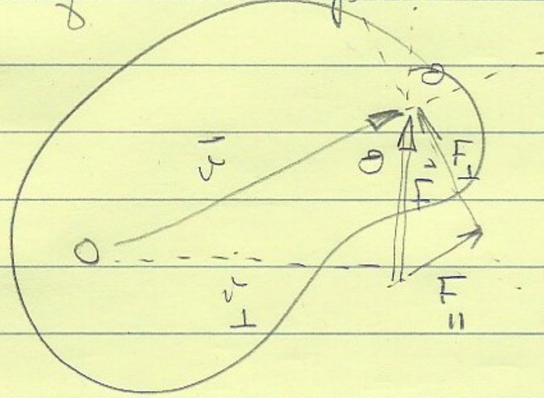
Summary

✓ Moment of Inertia is computed about a rotation point (axle).

$$I \equiv \int dm\, r^2$$



✓ Torque is computed about a rotation point



$$\text{torque} = F r \sin\theta = F r_\perp = F_\perp r$$

✓ Equation of motion

$$I \alpha = \tau$$