# The Corrected Diffusion Approximation

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# Summary

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  I model light propagation in biological tissues for the purpose of locating early stage cancer cells.

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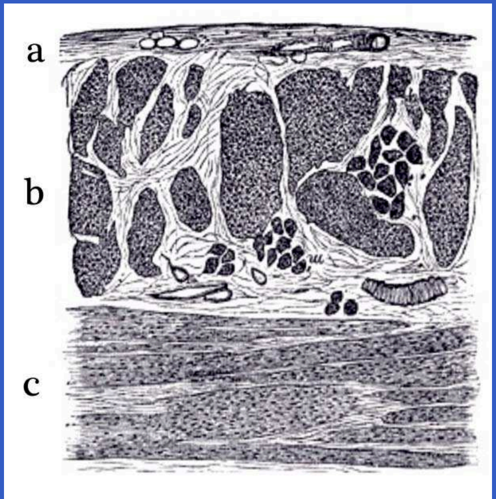
- Results

# Tissue Structure




a. Epithelial Layer

b. Stromal Layer

c. Smooth Muscle Layer

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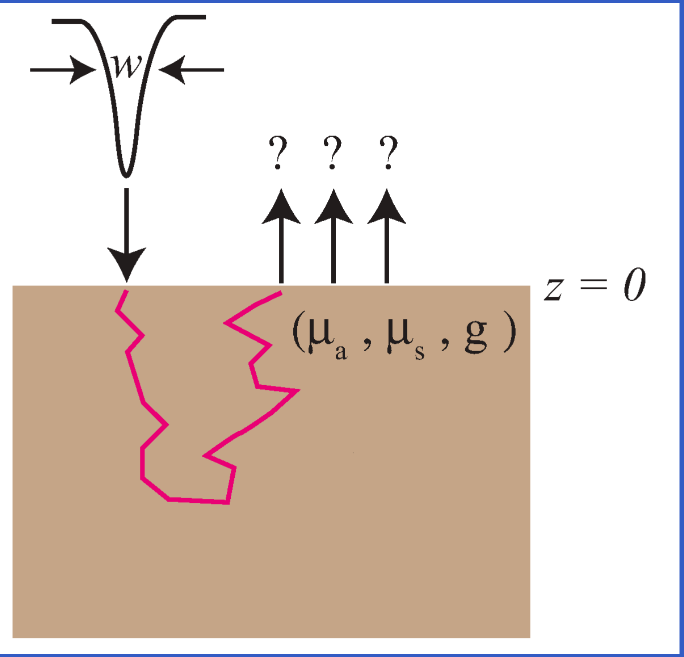
- Use a thin continuous beam incident normally on the medium
- Represent medium by a semi-infinite half space
- Given constant scattering and absorption coefficients



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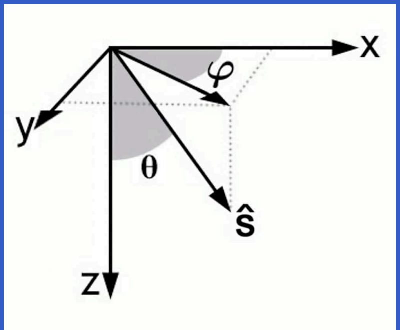
$p$ defines the fraction of power scattered in direction $\hat{\mathbf{s}}$ incident from direction $\hat{\mathbf{s}}'$.

$$\int_{\mathbb{S}^2} p(\hat{\mathbf{s}} \cdot \hat{\mathbf{s}}') d\hat{\mathbf{s}}' = 1$$



# Boundary Condition

For a normally incident, Gaussian beam we consider

$$I(\mu, \varphi, x, y, 0) - r(\mu) I(-\mu, \varphi, x, y, 0) = \frac{\delta(\mu - 1)}{2\pi} f(x, y), \quad 0 < \mu \leq 1$$

$$f(x, y) = \frac{1}{2\pi w^2} e^{-\dfrac{x^2 + y^2}{2w^2}}$$

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and

$$I \to 0 \quad \text{as} \quad z \to \infty$$

In this, $r(\mu)$ is the Fresnel reflection coefficient at the boundary.

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  The Diffusion Approximation is an approximation to the RTE.

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in this, $D = \dfrac{1}{3(\mu_a + \mu_s(1-g))}$, and $S$ is the interior source term.

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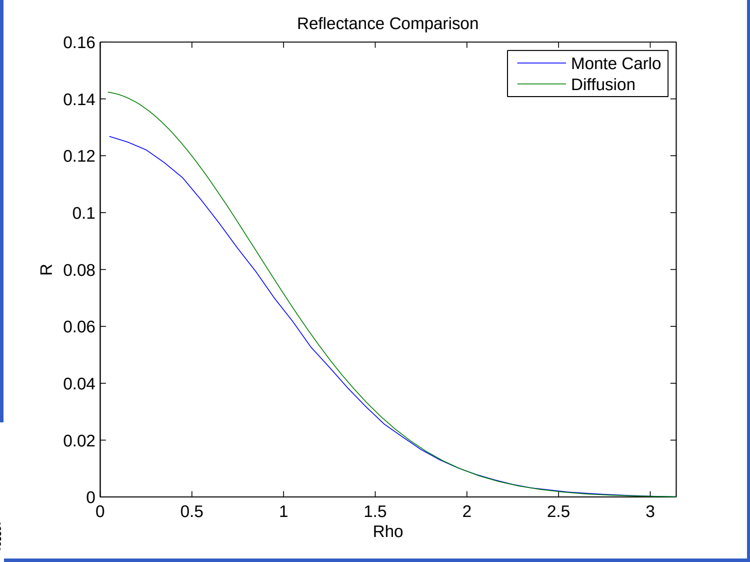
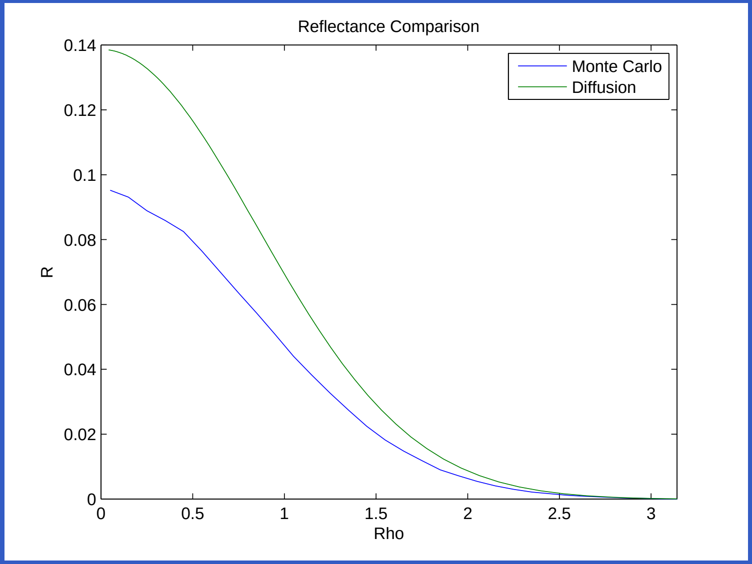
Problem: This is known to be invalid close to the source.



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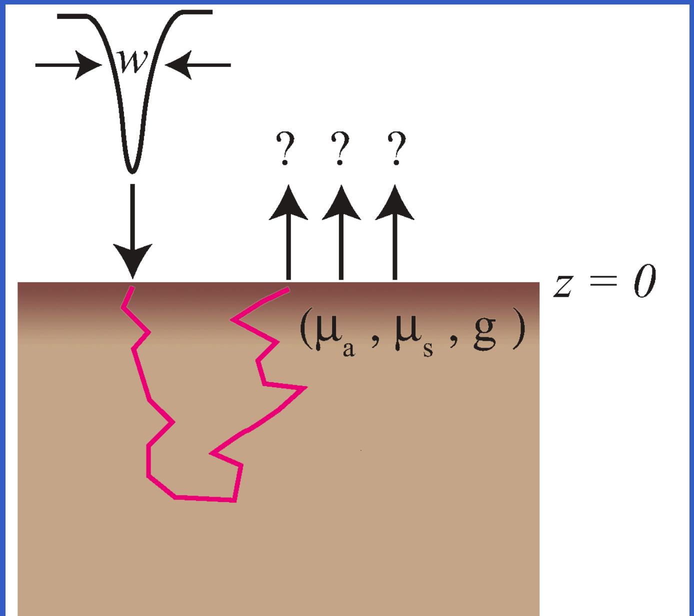
- Compute Interior Solution (Diffusion, $\Phi$)
- Compute Boundary Layer Solution (RTE, $\Psi$)
- Combine results to satisfy original conditions

$$I(x, y, z, \hat{\mathrm{s}}) = \Phi(x, y, z) + \Psi(x, y, z, \hat{\mathrm{s}})$$



# Derivation of CDA: Rescaling

- Three length scales in our analysis ($\ell_s \ll w \ll \ell_a$)

  Scattering mean free path, $\ell_s = \dfrac{1}{\mu_s}$

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- Use our length scales to define small parameters $\alpha$ and $\beta$

  $$\alpha = \frac{\ell_s}{\ell_a}$$

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  $$\alpha = \frac{\ell_s}{\ell_a}$$

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- Rescale $(x, y, z)$ with respect to $w$ which nondimensionalizes the problem
- Solve the rescaled, nondimensionalized equation using the fact that $\alpha \ll \beta \ll 1$

# CDA: Rescaled Problem

$$\beta\mu\partial_z I + \beta\sqrt{1-\mu^2}(\cos\varphi\partial_x I + \sin\varphi\partial_y I) + \alpha I + \mathscr{L} I = 0$$

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Subject to boundary conditions

$$I(\mu,\varphi,x,y,0) = \frac{\delta(\mu-1)}{2\pi} f(x,y) + r(\mu) I(-\mu,\varphi,x,y,0), \quad 0 < \mu \leq 1,$$

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In these, $r(\mu)$ is the Fresnel reflection coefficient at the boundary.

We represent $I$ as the sum of an interior solution and a boundary layer solution as in [a].

$(I = \Phi + \Psi)$

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[a]S. B. Rohde and A. D. Kim, “Modeling the diffuse reflectance due to a narrow beam incident on a turbid medium,” J. Opt. Soc. Am. A, **29**, 231-238 (2012).

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in this $\kappa = \dfrac{1}{3(1-g)}$, and we have a solution of the form

$$\Phi = \phi_0(\mathbf{r}) - \beta \hat{\mathbf{s}} \cdot [3\kappa \nabla \phi_0(\mathbf{r})] + O(\beta^2),$$

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$$\Phi = \phi_0(\mathbf{r}) - \beta \hat{\mathbf{s}} \cdot [3\kappa \nabla \phi_0(\mathbf{r})] + O(\beta^2),$$

This solution alone cannot satisfy the boundary condition, and we apply a boundary layer solution

# Boundary Layer Solution

Introduce stretched variable $z = \beta Z$, such that

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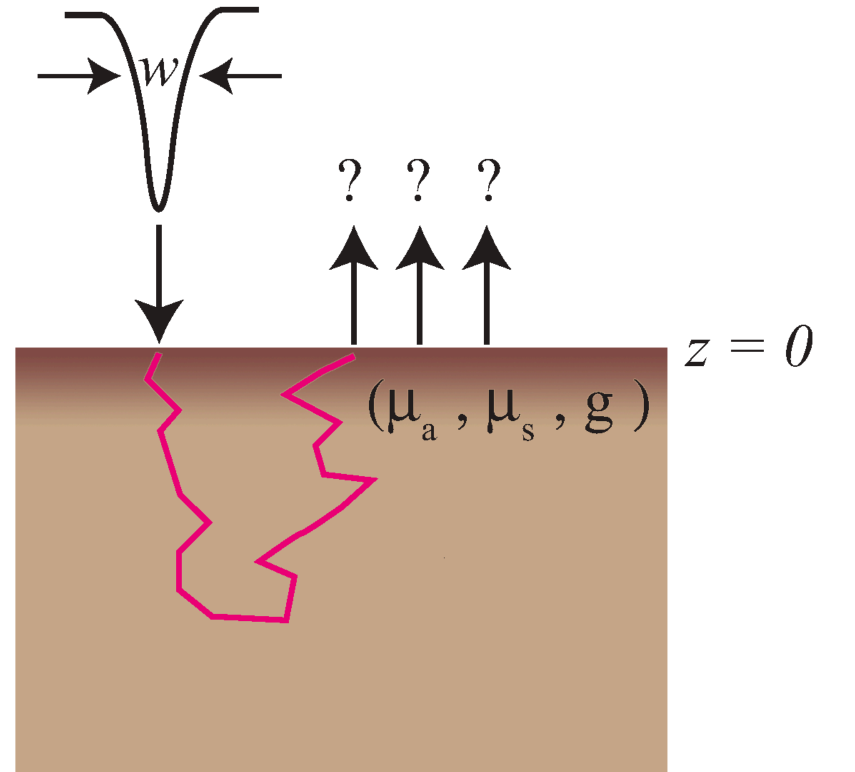
substitute into the RTE

$$\mu\psi_Z + \beta\sqrt{1-\mu^2}(\cos\varphi\psi_x + \sin\varphi\psi_y) + \alpha\psi + \mathscr{L}\psi = 0$$



# Boundary Condition

We apply the modified boundary condition for $\psi = I - \Phi$

$$\psi(\mu,\varphi,x,y,0) - r(\mu)\psi(-\mu,\varphi,x,y,0) =$$

$$\frac{\delta(\mu-1)}{2\pi} f(x,y) - [1-r(\mu)]\phi_0(x,y,0) + 3\beta\kappa\mu[1+r(\mu)]\phi_{0,z}(x,y,0), \quad 0 < \mu \leq 1$$

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Where $\psi = \psi_0 + \beta\psi_1$, and $\psi_0$ satisfies the 1-D RTE

$$\mu\psi_{0,Z} + \mathscr{L}\psi_0 = 0.$$

$\psi_1$ satisfies

$$\mu\psi_{1,Z} + \mathscr{L}\psi_1 = -\sqrt{1 - \mu^2}(\cos\varphi\psi_{0,x} + \sin\varphi\psi_{0,y})$$

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We next solve for $\phi$ and then apply the full boundary condition with the numerically calculated Green's function to determine $\psi$

# Interior Solution: Diffusion Approximation Solution

We can now solve

$$\nabla \cdot (\kappa \nabla \phi) - \alpha \phi = 0,$$

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Using Fourier Transforms $(x, y) \to (\xi, \eta)$

$$-\xi^2 \kappa \hat{\phi} - \eta^2 \kappa \hat{\phi} + \kappa \partial_z^2 \hat{\phi} - \alpha \hat{\phi} = 0,$$

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$$-\xi^2 \kappa \hat{\phi} - \eta^2 \kappa \hat{\phi} + \kappa \partial_z^2 \hat{\phi} - \alpha \hat{\phi} = 0,$$

Since $\phi$ decays exponentially in $z$ we set $\gamma(\xi, \eta) = -\sqrt{\alpha/\kappa + \xi^2 + \eta^2}$.
Substituting this into the BC we find

$$\hat{\phi} = \frac{f_0 \hat{f}(\xi, \eta)}{a_0 + b_0 \gamma}, \quad z = 0.$$

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# Reflectance Calculation

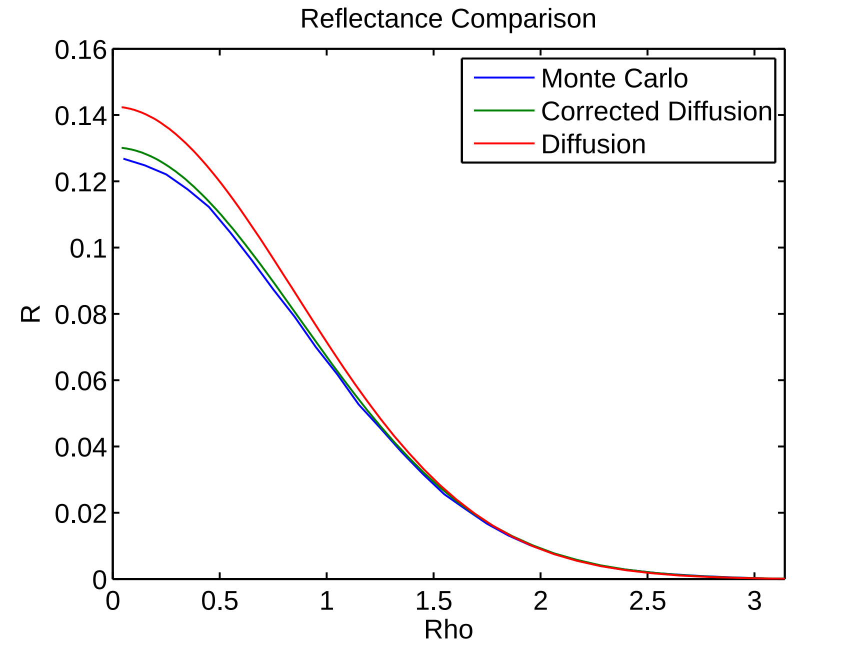
- Solve 1D RTE with Plane Wave Solutions [b]
- Build Greens Function numerically
- Integrate with our source terms to solve for $\Psi$ and $\Phi$, $I = \Psi + \Phi$
- Integrate over the range of angles exiting the medium to determine reflectance at the boundary

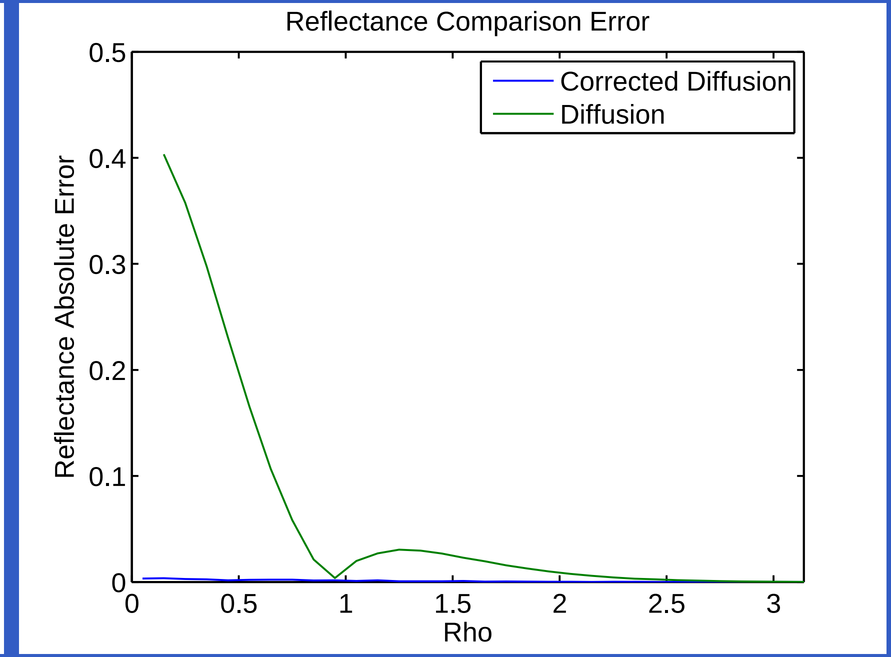
$$R(x, y) = -\iint_{NA} I(\mathbf{r}, \hat{\mathbf{s}})\hat{\mathbf{s}} \cdot \hat{\mathbf{z}} d\hat{\mathbf{s}}.$$

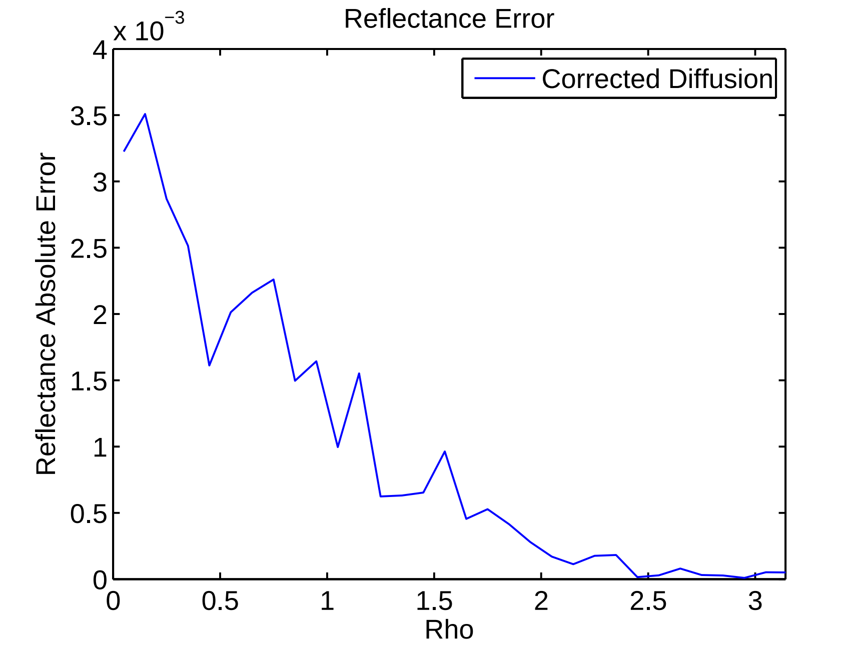
[b] A. D. Kim, “Correcting the diffusion approximation at the boundary,” J. Opt. Soc. Am. A **28**, 1007-1015 (2011).

# Results: How good is it?

$$\mu_a = .2(mm)^{-1},\ \mu_s = 100(mm)^{-1},\ g = 0.8,\ nrel = 1.4,\ BeamFWHM = 1$$



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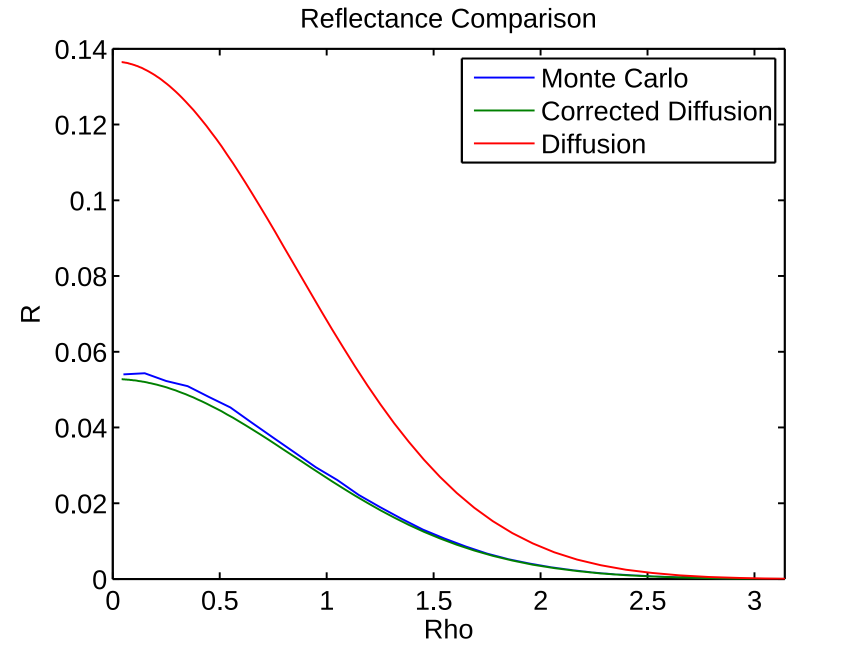


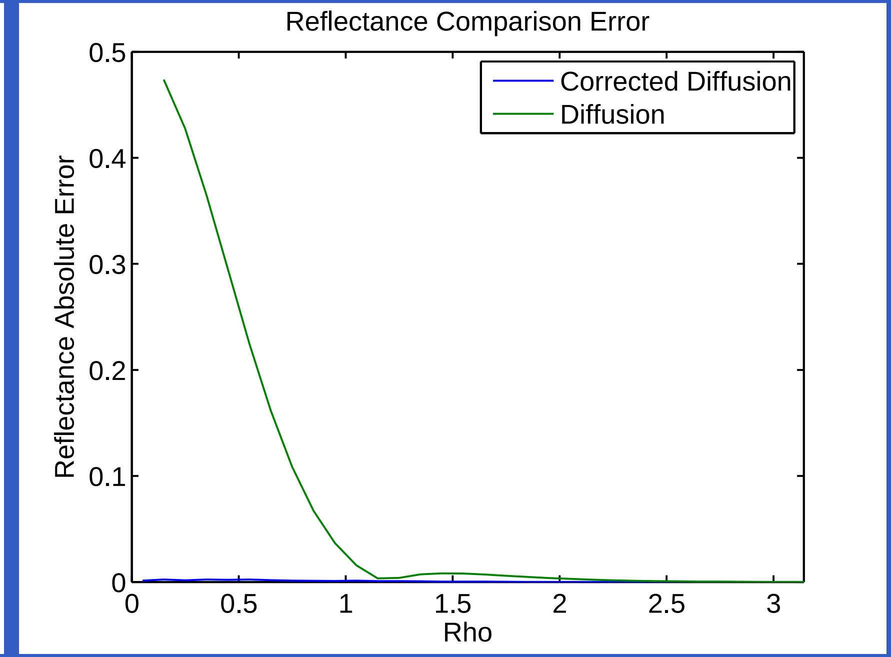


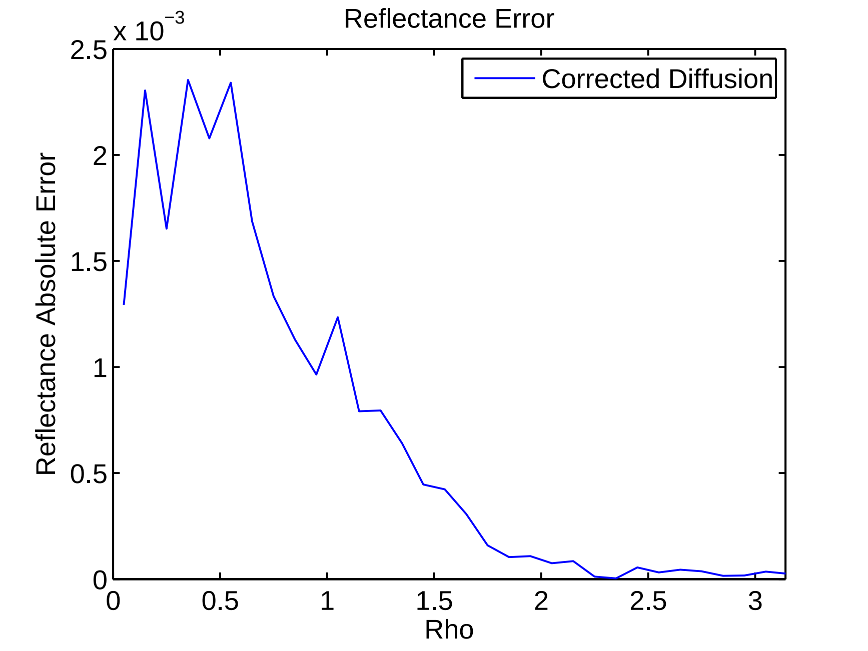

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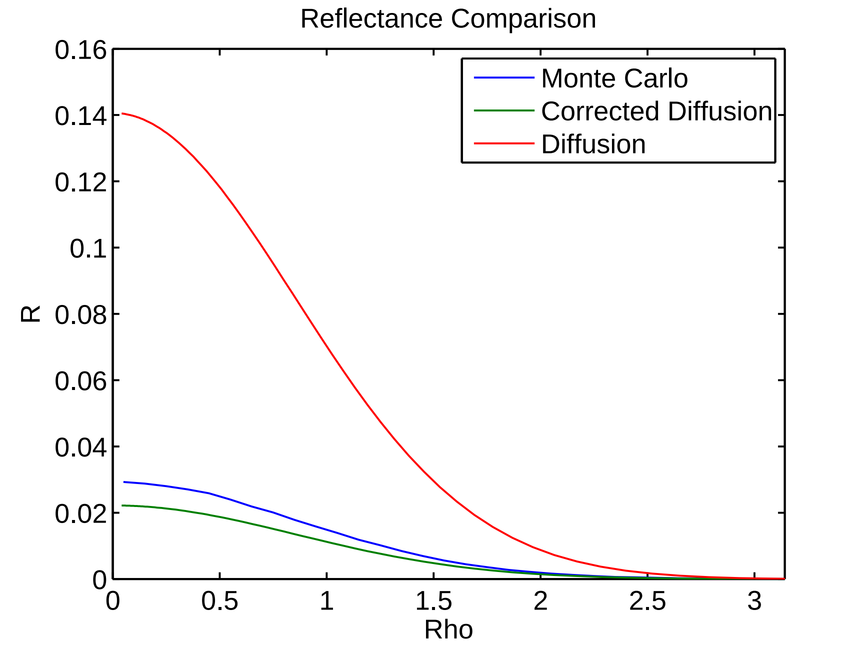
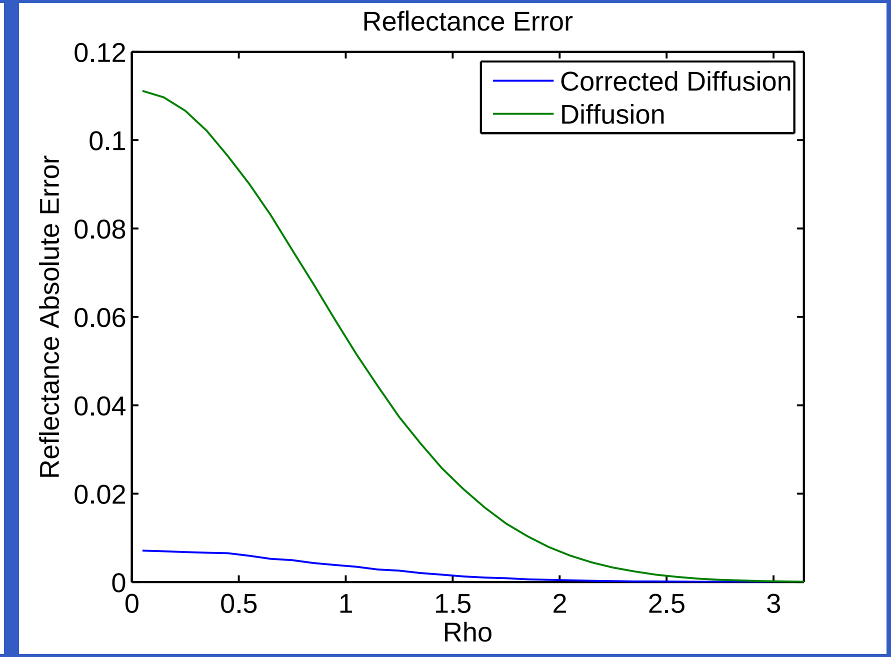
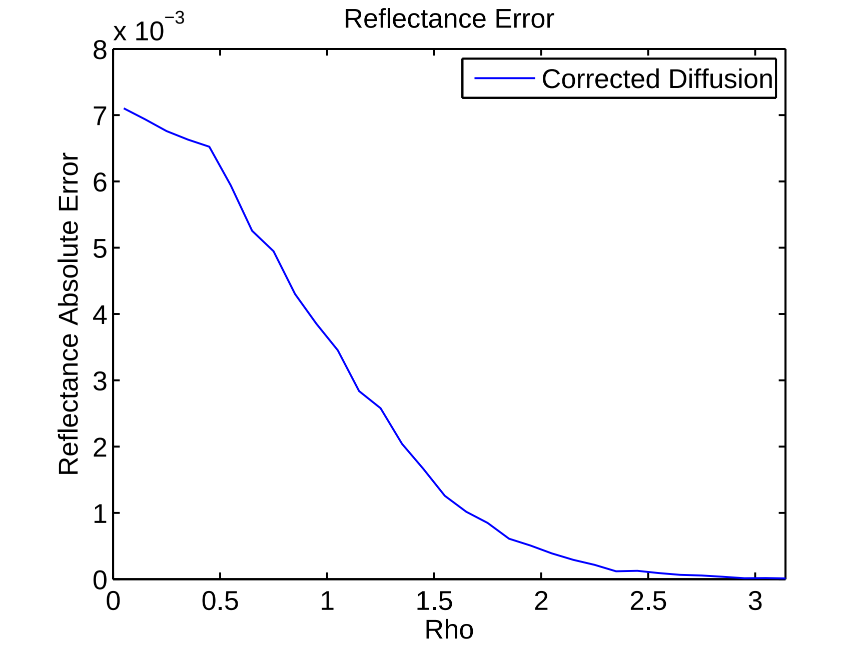
$$\mu_a = 5(mm)^{-1},\ \mu_s = 100(mm)^{-1},\ g = 0.8,\ nrel = 1.4,\ BeamFWHM = 1$$

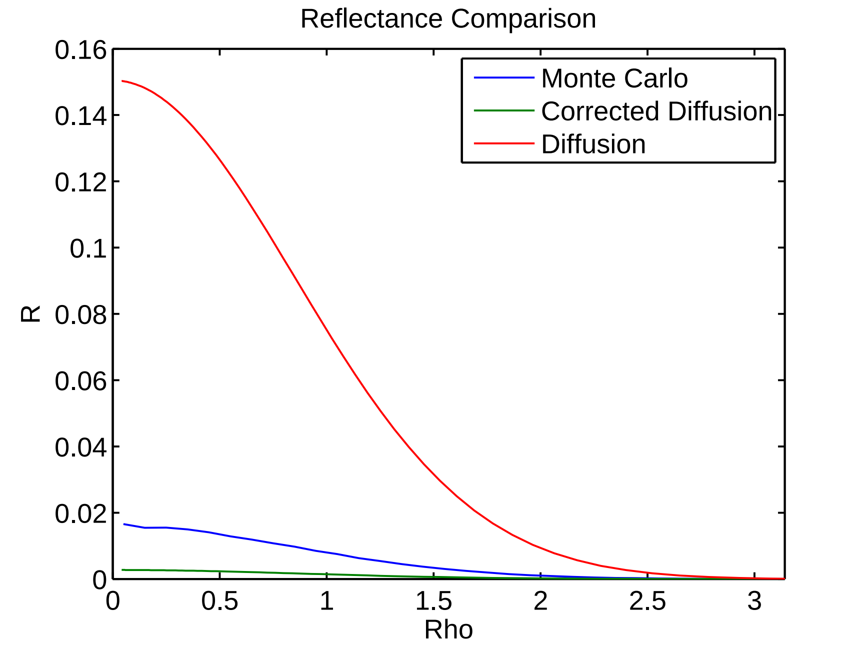


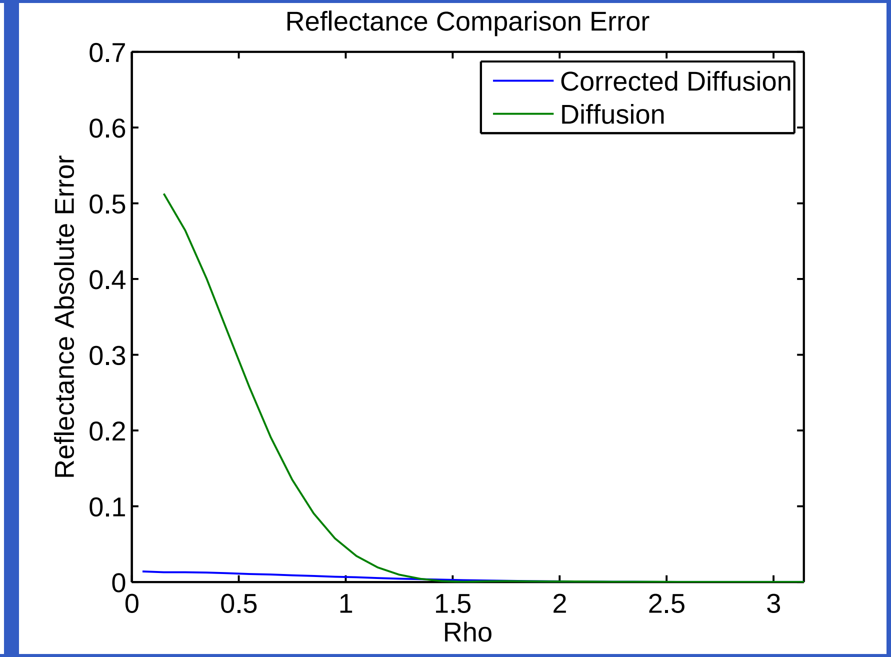



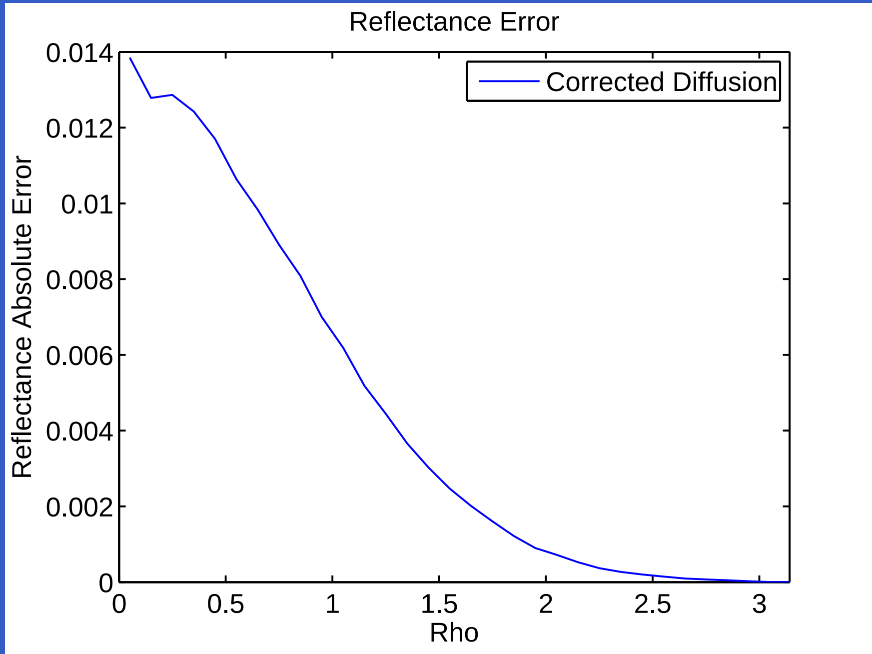



# Results: How good is it?

$$\mu_a = 10(mm)^{-1},\ \mu_s = 100(mm)^{-1},\ g = 0.8,\ nrel = 1.4,\ BeamFWHM = 1$$



# Conclusions and Acknowledgements

We constructed a forward model for accurate reflectance measurements close to the source

- We have extended it to include Fresnel reflection, layered tissues, and oblique incidence
- These models give us an option for modeling epithelial tissue specifically in an effort to locate early stage cancer cells, as well as an effective and invertible model for calculating optical properties of tissue

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Acknowledgements:

- Arnold D. Kim
- BLI team for the use of their Monte Carlo Virtual Tissue Simulator software
- A. D. Kim and S. Rohde acknowledge support from the National Science Foundation (NSF) for the work done on CDA.

# Thank you!

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