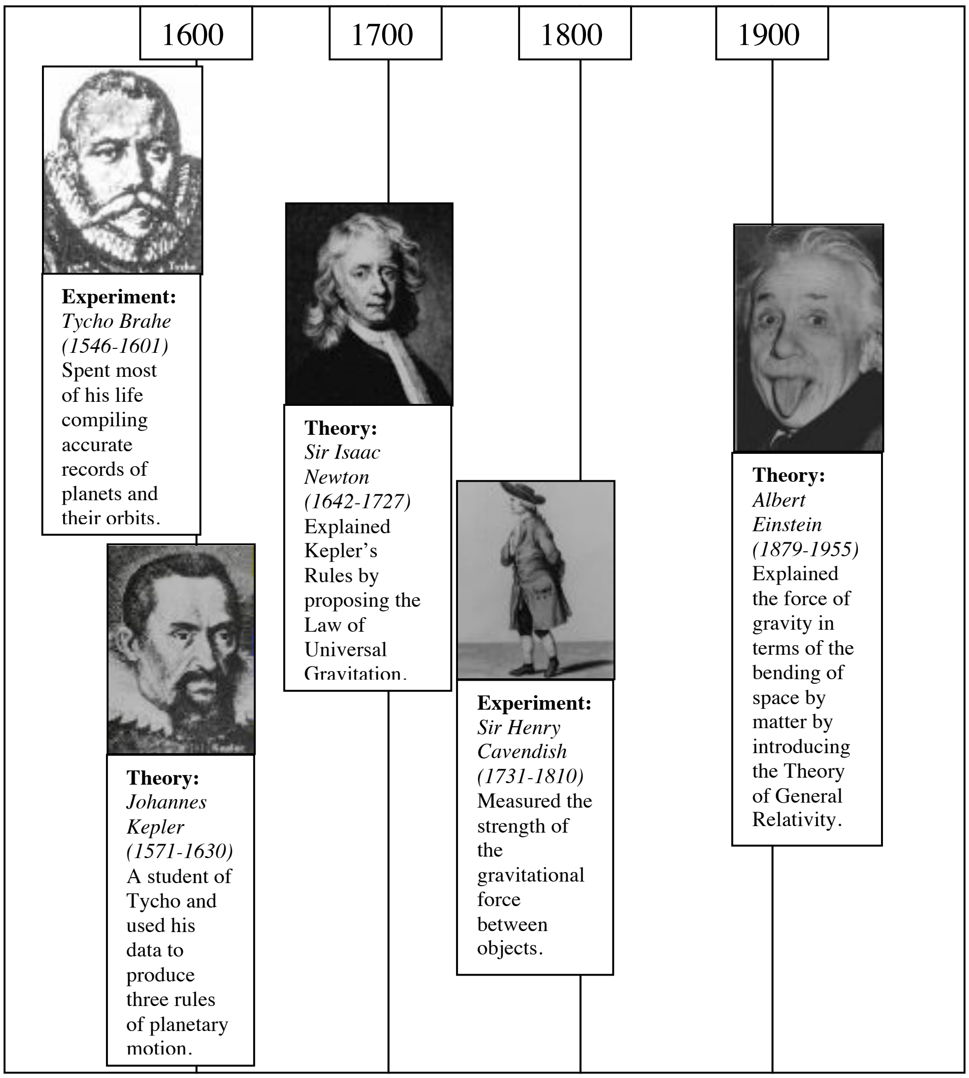
Section 39 – Gravitational Potential Energy & General Relativity

What is the universe made out of and how do the parts interact? We've learned that objects do what they do because of forces, energy, linear and angular momentum. In the last section, we continued to build an understanding of the theory of gravitation by following the development of the theory through time. Recall, Tycho provided good data for Kepler to invent his three rules which Newton's used as the basis of his Law of Universal Gravitation. This law was missing a numerical value for the gravitation constant, which the clever experiment of Cavendish revealed. In this section, we'll complete our journey by discussing Einstein's theory of gravitation, General Relativity.

So far our understanding of gravitational interactions has not included energy. We'll remedy this by reexamining gravitational potential energy in light of the Law of Gravitation.

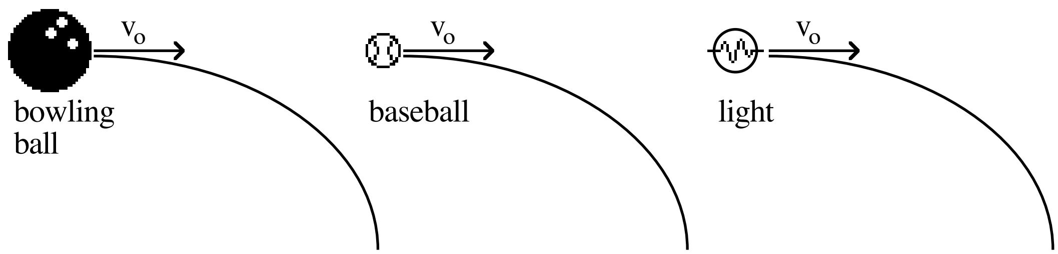


Section Outline

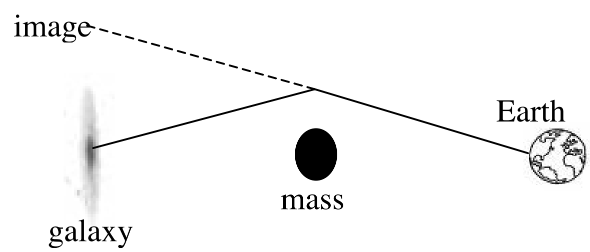
1. Einstein's Theory of Gravity
2. Gravitational Potential Energy

1. Einstein's Theory of Gravity

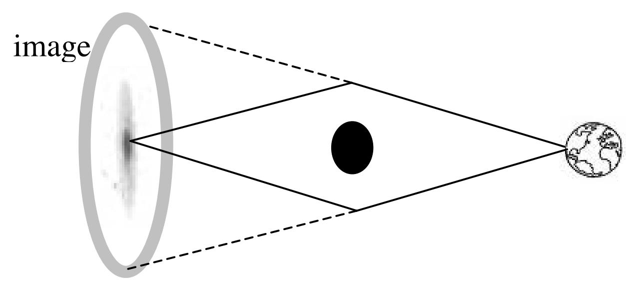
When the force of gravity is the only one acting, the mass of the moving object doesn't seem to matter. If a bowling ball and a baseball are given the same initial velocity, they execute the same motion. Einstein found this very bothersome because, he argued, it means that even massless objects like light should also be bent by gravity. He felt that this could only mean that the connection between gravity and mass is, in some sense, artificial.



We now have experimental evidence that Einstein's speculation about light being bent by gravity is actually correct. "Gravitational Lensing" is an example of light bent by gravity. Consider a distance galaxy that has a large mass between it and Earth. Light leaving the galaxy will be bent by the gravitation of the mass so the light will appear to be coming to Earth from a different direction as shown at the right.



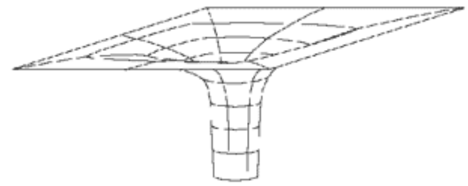
Light doesn't leave the galaxy along just one line. It leaves in all directions, so from Earth we would see the galaxy as a ring, as shown at the right. This is gravitational lensing."



At the right is a photograph from the Hubble Space Telescope Deep Field Camera. In it you will see many normal looking galaxies, but there are also many circular arcs of light. These are experimental evidence of gravitational lensing.

The reasoning behind Einstein's Theory of Gravity goes something like this:

1. Light is bent by gravity (experimentally verified).
2. Light always travels in "straight lines" because it always covers the shortest distance between two points in the least time (nothing travels faster than light).
3. The conclusion is that if the light is going straight, then space must be curved!
4. The amount of curvature is proportional to the mass.



This is the basis for the Theory of General Relativity. The idea of bent space is portrayed in the iconic image at the right.

So, the point of the Theory of General Relativity is that gravity isn't really a force. Instead, it is the result of the bending of space caused by massive objects. The trajectories of planets, satellites, and such is due to the bending of space and not the result of a force. For our purposes, we can still treat it like a force and we'll do so for the rest of the course.

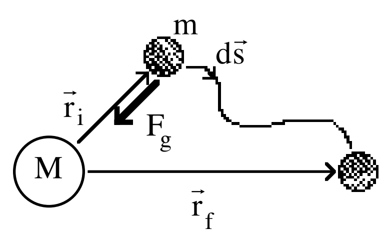
This completes our look at the history of the development of the understanding of gravitation. The interplay between theory and experiment known as the Scientific Method created our current knowledge of gravitation. The biggest steps are summarized below:

1. Tycho Brahe devoted his life to creating the most accurate experimental measurements of the positions of the planets.
2. Johannes Kepler, his student, used these measurements to design a theory summarized by Kepler's Three Rules of Planetary Motion.
3. Isaac Newton built upon Kepler's theory by explaining gravitation in terms of a force between massive objects, the Law of Universal Gravitation.
4. Henry Cavendish carried out a clever experiment to find the missing numerical value of Newton's theory called the gravitation constant.
5. Albert Einstein realized that the fact that the trajectory of objects due to gravitation never depends upon the mass of the object needed to be explained. His Theory of General Relativity provides this explanation.

There are other features of gravitation that still need to be explained by new theories and we'll talk about them in the next section.

2. Gravitational Potential Energy

We need to add energy to our understanding of the gravitational driven motion of objects. The gravitational potential energy can be found using the definition of potential energy,



$$\Delta U \equiv -W_c \Rightarrow \Delta U_g = -\int_{r_i}^{r_f} \vec{F}_g \bullet d\vec{s}$$

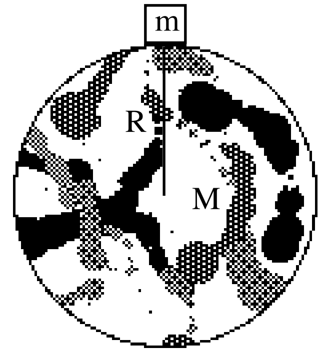
Since the gravitational force always points radially, $d\vec{s}$ and be replaced by $d\vec{r}$. Since we are moving out, the dot product gives a minus sign. Using the Law of Universal Gravitation, the integration can be done,

$$\Delta U_g = +\int_{r_i}^{r_f} F_g dr = GMm\int_{r_i}^{r_f} \frac{dr}{r^2} = GMm\left[-\frac{1}{r}\right]_{r_i}^{r_f} \Rightarrow U_{gf} - U_{gi} = -G\frac{Mm}{r_f} + G\frac{Mm}{r_i}.$$

By setting the initial potential energy to zero when the initial position is infinitely far away, we get the expression for the potential energy due to gravity,

The Gravitational Potential Energy $U_g = -G\dfrac{Mm}{r}$

Example 39.1: Find the gravitational potential energy of a 1.00kg mass at the surface of Earth.



Given: $M = 5.97x10^{24}$kg, m = 1.00kg, and $R = 6.40x10^6$m
Find: U_g = ?

Plugging the known values into the equation for gravitational potential energy,

$$U_g = -G\frac{Mm}{r} = -\left(6.67x10^{-11}\right)\frac{(5.97x10^{24})(1.00)}{6.40x10^6} \Rightarrow$$
$$\boxed{U_g = -6.22x10^7 J}.$$

What does the minus sign mean? The potential energy grows (gets less negative) as the mass is moved further away. When the mass is infinitely far away the potential energy is zero. The minus sign indicates that the mass, m, is "bound" to the Earth.

What happened to our old friend U_g = mgh? Suppose we lift the mass up a distance, h. Let's find the change in gravitational potential energy. The distance between the center of Earth and the 1kg mass is initially just the radius of Earth, so

$$U_{gi} = -G\frac{Mm}{R}.$$

The final distance is the radius of Earth plus the height h.

$$U_{gf} = -G\frac{Mm}{R+h}.$$

The change in potential energy is,

$$\Delta U_g = U_{gf} - U_{gi} = -G\frac{Mm}{R+h} + G\frac{Mm}{R} = GMm\left\{\frac{-1}{R+h} + \frac{1}{R}\right\} = GMm\left\{\frac{-R+R+h}{R(R+h)}\right\} = \frac{GMmh}{R(R+h)}.$$

The h in the denominator can be neglected in comparison to R because even a few hundred meters is much smaller than the radius of Earth. Now,

$$\Delta U_g \approx \frac{GMmh}{R^2} = \left(\frac{GMm}{R^2}\right)h.$$

The quantity in parentheses is just the force on the mass at the surface of Earth. We usually write that as,

$$F_g = \frac{GMm}{R^2} = mg.$$

Finally, the change in the gravitational potential energy can be written as,

$$\Delta U_g = \left(\frac{GMm}{R^2}\right)h = mgh.$$

So, for small distances from Earth (small compared to Earth's radius) mgh works great. This really amounts to a choice of where the zero for potential energy is established. The expression for gravitational potential energy assumes the zero is at infinity while mgh uses a zero near the surface of Earth.

Example 39.2: Find the minimum velocity at which a rocket must leave Earth in order to "escape" Earth's gravity.

Given: $M = 5.97\text{x}10^{24}\text{kg}$ and $R = 6.40\text{x}10^{6}\text{m}$
Find: $v_o = ?$

$$K_o = \tfrac{1}{2}mv_o^2 \qquad\qquad K = \tfrac{1}{2}mv^2$$
$$U_o = -G\frac{Mm}{R} \qquad\qquad U = -G\frac{Mm}{r} \approx 0$$

Applying the Law of Conservation of Energy,

$$\Delta K + \Delta U = 0 \Rightarrow \left(\tfrac{1}{2}mv^2 - \tfrac{1}{2}mv_o^2\right) + \left(0 - G\frac{Mm}{R}\right) = 0 \Rightarrow \tfrac{1}{2}mv^2 - \tfrac{1}{2}mv_o^2 = G\frac{Mm}{R}.$$

Solving for the initial velocity,

$$v_o = \sqrt{v^2 + \frac{2GM}{R}}.$$

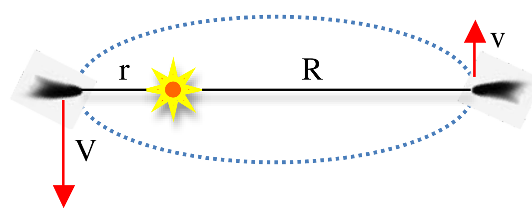
The minimum will occur when the final velocity is zero. This initial velocity is called the "escape velocity."

$$v_e = \sqrt{\frac{2GM}{R}} = \sqrt{\frac{2(6.67\text{x}10^{-11})(5.98\text{x}10^{24})}{6.37\text{x}10^{6}}} \Rightarrow \boxed{v_e = 11.2\text{km/s}}.$$

Kinetic energy must be added to make the total energy greater than zero. If the total energy of a system is less than zero it is a "bound system."

Example 39.3: Let's look at the Halley's comet example again. Recall, it is closest distance to the sun is 0.586AU and the farthest distance is 35.1AU. When it is farthest away it is traveling about 0.911km/s. Find the speed when it is closest to the sun using energy methods. Compare this answer with the answer from the Law of Conservation of Angular Momentum.

Given: $r = 0.586\text{AU} = 8.77\times10^{10}\text{m}$,
$R = 35.1\text{AU} = 5.25\times10^{12}\text{m}$, $M = 1.99\times10^{30}\text{kg}$ and
$v = 911\text{m/s}$.
Find: $V = ?$



At the farthest distance the total energy is,

$$E_a = \tfrac{1}{2}mv^2 - G\tfrac{mM}{R}.$$

At the closest distance the energy is,

$$E_p = \tfrac{1}{2} mV^2 - G\tfrac{mM}{r}.$$

Using the Law of Conservation of Energy,

$$E_a = E_p \Rightarrow \tfrac{1}{2} mv^2 - G\tfrac{mM}{R} = \tfrac{1}{2} mV^2 - G\tfrac{mM}{r}.$$

Solving for the speed at closest approach,

$$V = \sqrt{v^2 + 2GM\left(\tfrac{1}{r} - \tfrac{1}{R}\right)}.$$

Putting in the numbers,

$$V = \sqrt{(911)^2 + 2(6.67x10^{-11})(1.99x10^{30})\left(\tfrac{1}{8.77x10^{10}} - \tfrac{1}{5.25x10^{12}}\right)} \Rightarrow \boxed{V = 54.6\tfrac{km}{s}}.$$

In agreement with our earlier results.

Section Summary

We completed our historical trip through the time to see the progression of our understanding of gravity by looking at Einstein's Theory of General Relativity. Gravity is not so much a force as the cause of the bending of space. This bending of space is responsible for gravitational lensing.

Summarizing the history:

1. Tycho Brahe devoted his life to creating the most accurate experimental measurements of the positions of the planets.
2. Johannes Kepler, his student, used these measurements to design a theory summarized by Kepler's Three Rules of Planetary Motion.
3. Isaac Newton built upon Kepler's theory by explaining gravitation in terms of a force between massive objects, the Law of Universal Gravitation.
4. Henry Cavendish carried out a clever experiment to find the missing numerical value of Newton's theory called the gravitation constant.
5. Albert Einstein realized that the fact that the trajectory of objects due to gravitation never depends upon the mass of the object needed to be explained. His Theory of General Relativity provides this explanation.

We then used the Law of Gravitation to find

The Gravitational Potential Energy $U_g = -G\dfrac{Mm}{r}$.