Chapter 23 - Electric Fields

Problem Set #2 - due:
Ch 23 - 1, 2, 5, 8, 14, 19, 26, 30, 31, 35, 38, 39, 50, 58, 61, 62

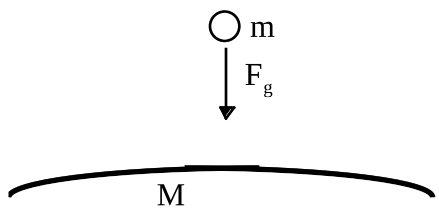
The motion of an electric charge is determined by the forces that act on it. The forces it feels that are exerted by all the other charges can be described in terms of the electric field due to the other charges.

Lecture Outline
1. The Definition of Electric Field
2. Electric Field Lines
3. The Electric Field Due to Point Charges
4. The Electric Field Due to Continuous Charge Distributions
5. The Force on Charges in Electric Fields

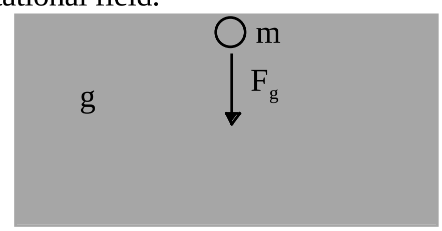
1. The Definition of Electric Field

The idea of a field is required in order to explain the "action at a distance."

Recall the idea of gravitational field.

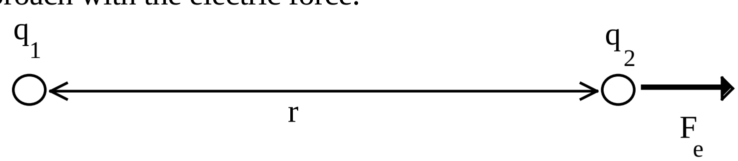


In this view, Earth creates a force on the mass m. This is "insane." Earth isn't even touching the mass. So we introduce the idea of a gravitational field.

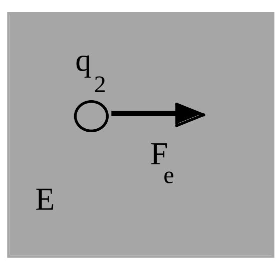


Now we take the view that the field due to Earth, g, is exerting the gravitational force on the mass. Since the force is $\vec{F}_g = m\vec{g}$, the gravitational field is defined as, $\vec{g} \equiv \frac{\vec{F}_g}{m}$.

Let's take the same approach with the electric force.



Instead of thinking of q_1 exerting the force on q_2, we think of q_1 creating a field and the field exerting the force on q_2.



Mathematically, we can write $\vec{F}_{21} = q_2 \vec{E}_1 \qquad \vec{E}_1 \equiv \frac{\vec{F}_{21}}{q_2}$ which is the definition of the electric field.

$$\boxed{\vec{E} \equiv \frac{\vec{F}}{q} \qquad \text{The Definition of Electric Field}}$$

Note that the field is the force felt by each Coulomb of charge.

2. Electric Field Lines

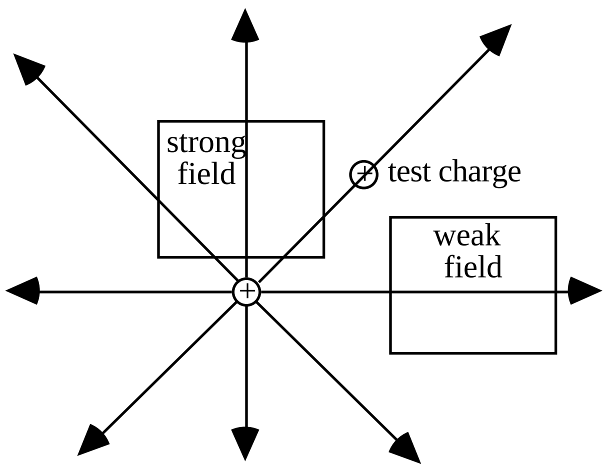
We need a more descriptive image of the field. The most useful idea is “Electric Field Lines.”

Beyond the Mechanical Universe (vol. 29 ch 11 & 12)

Field Lines or Lines of Force are used to visualize the field. The rules for drawing them are:

1. The tangent to the field line points in the direction of the force on a positive test charge.
2. The density of lines is proportional to the strength of the field (bigger charges make more lines).

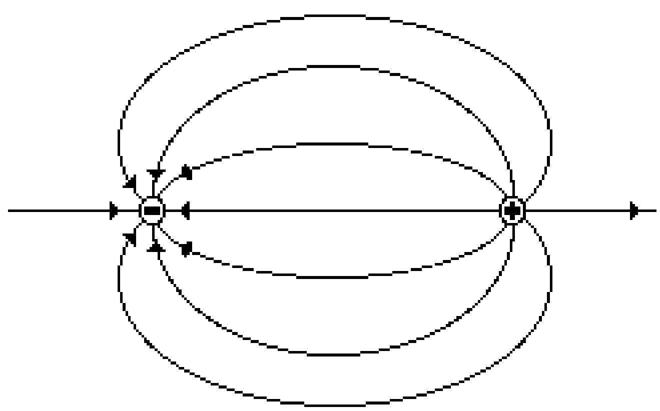
Example 1: Sketch the field due to a positive point charge.



Example 2: Sketch the field due to a dipole.

A dipole consists of two equal and opposite charges.

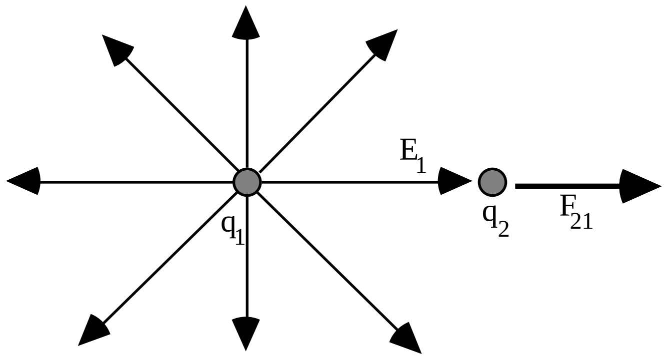
Notice that all the field lines that leave the positive charge end on the negative charge.



Beyond the Mechanical Universe (vol. 29 ch 14)

3. The Electric Field Due to Point Charges

We have now reduced the problem of figuring out what a charge will do, to a problem of finding the field that it feels. The next step is to develop techniques to calculate the fields created by charges.



Starting with the definition of the electric field

$$\vec{E}_1 \equiv \frac{\vec{F}_{21}}{q_2}.$$

Using Coulomb's Rule, $\vec{F}_e = k\frac{q_1 q_2}{r^2}\hat{r}$, we get the electric field caused by the point charge, q_1,

$$\vec{E}_1 = \frac{k\frac{q_1 q_2}{r^2}\hat{r}}{q_2} = k\frac{q_1}{r^2}\hat{r}$$

$$\boxed{\vec{E} = k\frac{q}{r^2}\hat{r} \qquad \text{E-Field of a Point Charge}}$$

Example 3: Calculate the electric field felt by the electron in a hydrogen atom.

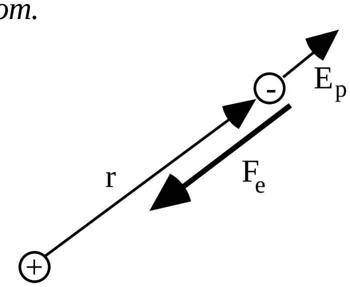
$r = 0.052\text{nm}$ and $q_p = 1.6 \times 10^{-19}\text{C}$

Using the definition of electric field,

$$\vec{E} \equiv k\frac{q}{r^2}\hat{r} \qquad \vec{E}_p \equiv k\frac{q_p}{r^2}\hat{r}$$

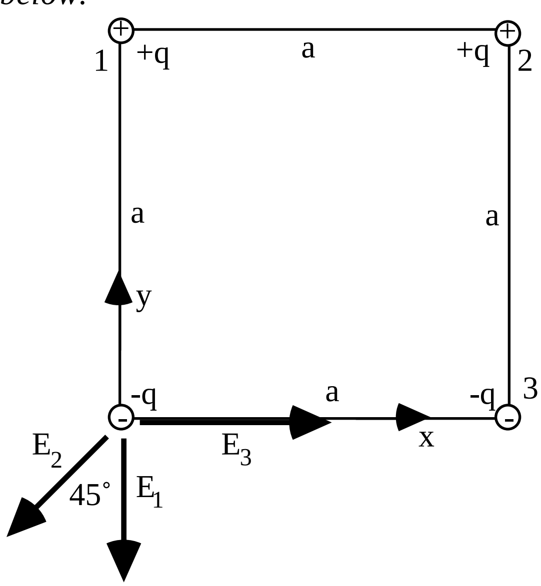
$E_p = 5.3 \times 10^{11}\text{N/C}$

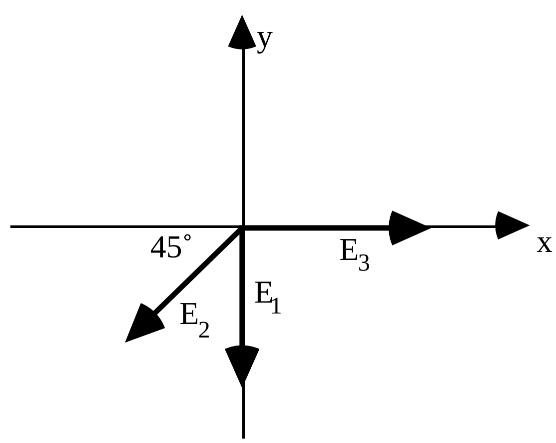
Notice the directions of $\vec{E}_p$ and $\vec{F}_e$.



For a collection of point charges, $\vec{E} = \vec{E}_1 + \vec{E}_2 + \cdots + \vec{E}_N$. Note that this is a vector sum.

Example 4: Find the electric field felt by the charge on the bottom-left corner of the square shown below.





Using the field due to a point charge, $\vec{E} = k\frac{q}{r^2}\hat{r}$, find the magnitude of each field vector.

$E_1 = k\dfrac{q}{a^2}$, $E_2 = k\dfrac{q}{2a^2} = \dfrac{E_1}{2}$, $E_3 = k\dfrac{q}{a^2} = E_1$.

Adding up the vector components,

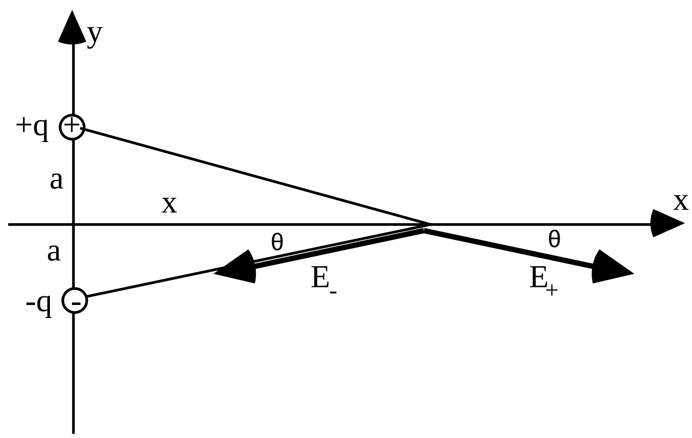
$$E_x = E_3 - E_2\cos\theta = E_1 - E_1\frac{1}{2\sqrt{2}} = E_1\left(1 - \frac{1}{2\sqrt{2}}\right)$$

$$E_y = -E_1 - E_2\sin\theta = -E_1 - E_1\frac{1}{2\sqrt{2}} = -E_1\left(1 + \frac{1}{2\sqrt{2}}\right)$$

$\vec{E} = E_1\left\{\left(1 - \dfrac{1}{2\sqrt{2}}\right)\hat{i} - \left(1 + \dfrac{1}{2\sqrt{2}}\right)\hat{j}\right\}$ or in terms of magnitude and direction ,

$$E = E_1\sqrt{\left(1 - \frac{1}{2\sqrt{2}}\right)^2 + \left(1 + \frac{1}{2\sqrt{2}}\right)^2} = \frac{3}{2}E_1 \text{ and } \theta = \arctan\frac{-\left(1 + \frac{1}{2\sqrt{2}}\right)}{\left(1 - \frac{1}{2\sqrt{2}}\right)} = -64^\circ \text{ from the } x\text{-axis}$$

<u>*Example 5:*</u> *Find the electric field due to a dipole along the perpendicular bisector of its axis.*



Use the field due to a point charge, $\vec{E} = k\dfrac{q}{r^2}\hat{r}$, to find the magnitude of both field vectors. Notice they are the same magnitude, $E_+ = E_- = k\dfrac{q}{a^2 + x^2}$.

Adding up the vector components,

$$E_x = E_+\cos\theta - E_-\cos\theta = 0$$

$$E_y = -E_+\sin\theta - E_-\sin\theta = -2E_+\sin\theta = -2k\frac{q}{a^2 + x^2}\frac{a}{\sqrt{a^2 + x^2}}$$

$\vec{E} = -k\dfrac{2qa}{\left(x^2 + a^2\right)^{\frac{3}{2}}}\hat{j} = -k\dfrac{p}{\left(x^2 + a^2\right)^{\frac{3}{2}}}\hat{j}$ where $p \equiv 2qa$ is the "dipole moment."

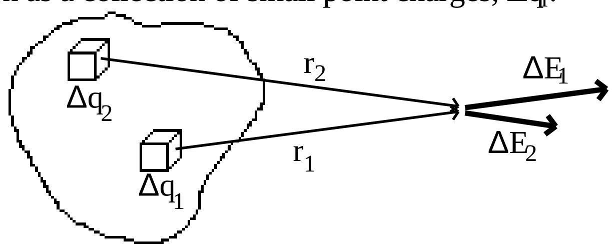
Check the limits as $x \to 0$ and $x \to \infty$.

$$\boxed{\vec{p} \equiv q\vec{d} \qquad \text{The Definition of Electric Dipole Moment}}$$

where $\vec{d}$ is the displacement vector from the negative charge to the positive charge.

4. The Electric Field Due to Continuous Charge Distributions

Consider a charge distribution as a collection of small point charges, Δq_i.



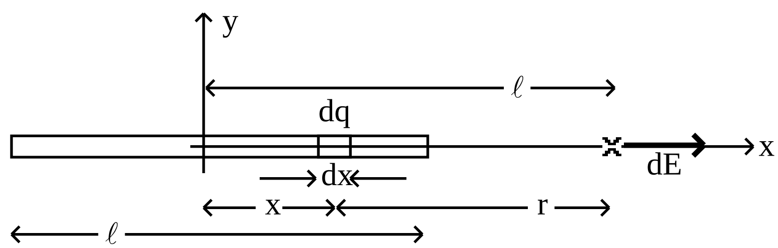
The total electric field is just the sum of the fields of the small (point) charges Δq's.

$$\vec{E} = \sum \Delta\vec{E}_i = \sum k\frac{\Delta q_i}{r_i^2}\hat{r}_i$$

For maximum accuracy we want the Δq's to become smaller and smaller. In this limit, the sum becomes an integral.

$$\boxed{\vec{E} = k\int\frac{dq}{r^2}\hat{r} \qquad \text{E-Field of a Continuous Charge Distribution}}$$

Example 6: Find the electric field due to a line of total charge, q, and length, ℓ, at a distance, ℓ, from the center of the line (a)along the line and (b)perpendicular from the center.



(a)

We need the field due to a continuous charge distribution, $\vec{E} = k\int\frac{dq}{r^2}\hat{r}$

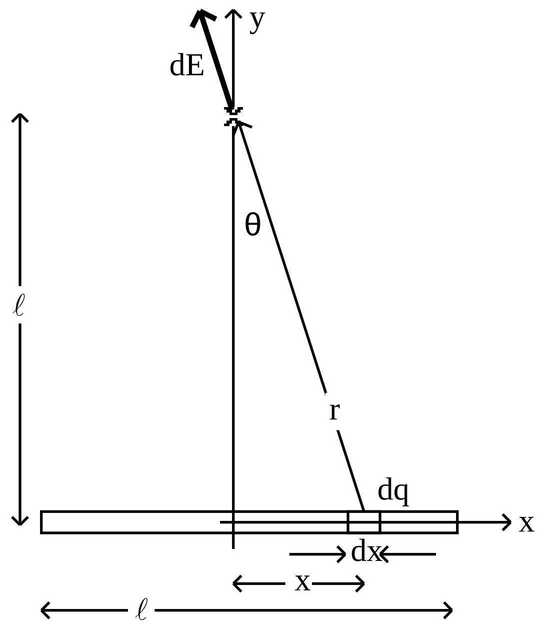
The field is along the x-direction so we can ignore the vector signs. The charge in a small segment, dq, is to the total charge, q, as the length of the segment, dx, is to the total length, ℓ,

$$\frac{dq}{q} = \frac{dx}{\ell} \qquad dq = \frac{q}{\ell}dx$$

The distance between the charge element, dq, and the point where we want the field is, $r = \ell - x$.

$$E = k\int_{-\frac{\ell}{2}}^{\frac{\ell}{2}}\frac{q}{\ell}\,\frac{dx}{(\ell - x)^2} \qquad \text{let } u = \ell - x \text{ then } du = -dx$$

$$E = -k\frac{q}{\ell}\int u^{-2}du = k\frac{q}{\ell}\left[\frac{1}{u}\right] = k\frac{q}{\ell}\left[\frac{1}{\ell - x}\right]_{-\frac{\ell}{2}}^{+\frac{\ell}{2}} = k\frac{q}{\ell}\left\{\frac{1}{\frac{\ell}{2}} - \frac{1}{\frac{3\ell}{2}}\right\} = k\frac{q}{\ell}\left\{\frac{2}{\ell} - \frac{2}{3\ell}\right\} = \frac{4}{3}k\frac{q}{\ell^2}$$



(b)Using the field due to the point charge dq, $d\vec{E} = k\frac{dq}{r^2}\hat{r}$. The symmetry of this problem requires that the x components will cancel out leaving only the y components, $dE_y = k\frac{dq}{r^2}\cos\theta$.

Again, the charge in a small segment, dq, is to the total charge, q, as the length of the segment, dx, is to the total length, ℓ,

$$\frac{dq}{q} = \frac{dx}{\ell} \qquad dq = \frac{q}{\ell}dx$$

The distance between the charge element, dq, and the point where we want the field is, $r = \sqrt{\ell^2 + x^2}$.

$$dE_y = k\frac{q}{\ell}\frac{dx}{\ell^2 + x^2}\cos\theta$$

Now we need to express x in terms of θ to complete the integration.

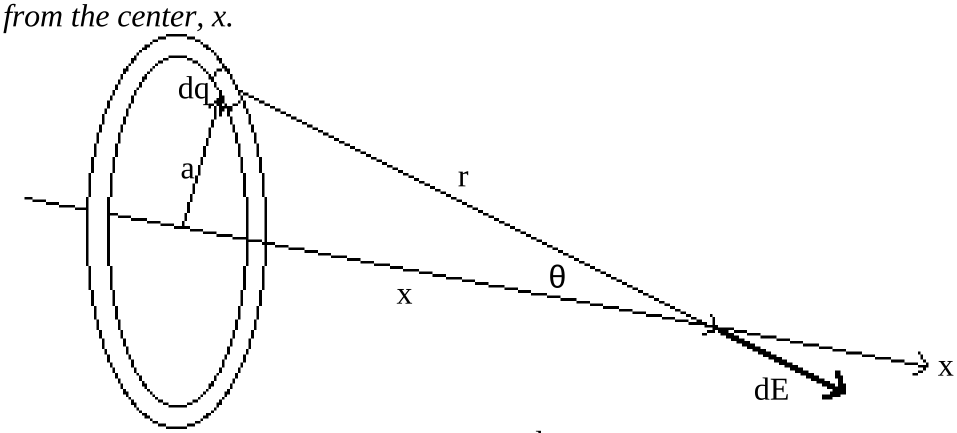
$$x = \ell\tan\theta \qquad \begin{cases} \ell^2 + x^2 = \ell^2\left(1 + \tan^2\theta\right) = \ell^2\sec^2\theta \\ dx = \ell\sec^2\theta\, d\theta \end{cases}$$

$$dE_y = k\frac{q}{\ell}\frac{\ell\sec^2\theta\, d\theta}{\ell^2\sec^2\theta}\cos\theta = k\frac{q}{\ell^2}\cos\theta\, d\theta$$

Now we can add up the contributions from all the other dq's by integrating over the angle,

$$E_y = k\frac{q}{\ell^2}\int_{-\theta_o}^{+\theta_o}\cos\theta\, d\theta = 2k\frac{q}{\ell^2}\sin\theta_o \text{ where } \sin\theta_0 = \frac{1}{\sqrt{5}} \qquad \underline{\underline{E_y = \frac{2}{\sqrt{5}}k\frac{q}{\ell^2}}}.$$

<u>Example 7:</u> Find the field on the axis of a ring of charge, q, and radius, a, as a function of the distance from the center, x.



Using the field due to the point charge dq, $d\vec{E} = k\frac{dq}{r^2}\hat{r}$. By the symmetry of this problem the field will only point in the x-direction so we only have to worry about the x-components,

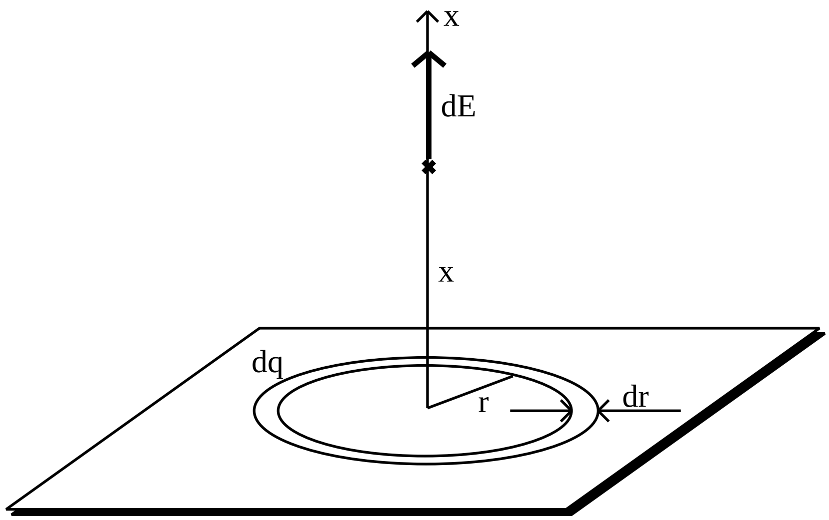
$dE_x = k\frac{dq}{r^2}\cos\theta$. Notice that r is the same for all the dq's and $\cos\theta$ is also the same for all dq's so the integral is straight forward.

$$E_x = k\int\frac{dq}{r^2}\cos\theta = k\frac{\cos\theta}{r^2}\int dq = k\frac{q}{r^2}\cos\theta$$

Note that $r^2 = a^2 + x^2$ and $\cos\theta = \frac{x}{\sqrt{a^2+x^2}}$ so we can write $E_x = k\frac{q\ x}{\left(a^2+x^2\right)^{\frac{3}{2}}}$.

Check the answer by examining the limits as x goes to zero and to infinity.

<u>*Example 8:*</u> *Find the electric field a distance, x, away from an infinite plate of charge density, $\sigma(^C/_{m^2})$.*



From example 10 we know the field due to a ring of charge, so break the plane up into an infinite number of rings with charge, dq, radius, r, and thickness dr. Each ring creates a field, $dE = k\frac{x\ dq}{\left(r^2+x^2\right)^{3/2}}$.

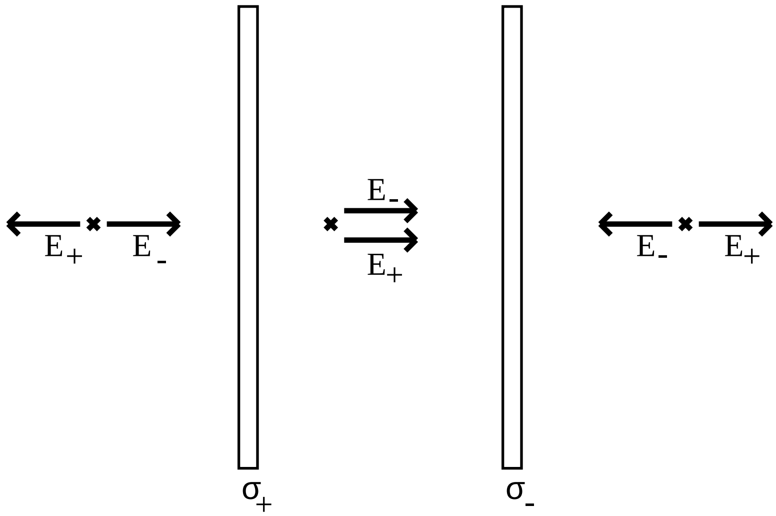
The charge on each ring can be found by multiplying the charge density by the area of the ring $dq = \sigma 2\pi r\ dr$.

$$dE = k\frac{x\ \sigma 2\pi r\ dr}{\left(r^2+x^2\right)^{3/2}} \qquad E = 2\pi\sigma kx\int_0^\infty \frac{r\ dr}{\left(r^2+x^2\right)^{3/2}}.$$

The integration can be completed by letting $u = r^2 + x^2 \qquad du = 2r\ dr$

$$E = \pi\sigma kx\int_{x^2}^\infty u^{-3/2}du = \pi\sigma kx\left[\frac{-2}{u^{1/2}}\right]_{x^2}^\infty = \pi\sigma kx\frac{2}{x} = \underline{\underline{2\pi\sigma k}}$$ This is a constant!

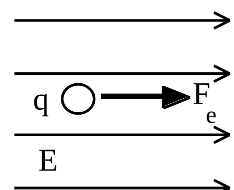
<u>*Example 9:*</u> *Find the electric field inside and outside two large, parallel, equal but oppositely charged plates.*



From example 8 we know the field due to a parallel plate. We just need to add the fields from each plate.

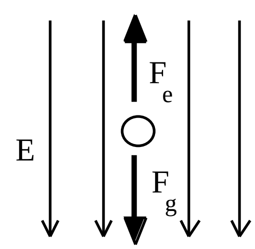
$$E_{in} = E_+ + E_- = 2\pi\sigma k + 2\pi\sigma k = 4\pi\sigma k = \underline{\underline{\frac{\sigma}{\varepsilon_o}}} \quad \text{and } E_{out} = E_+ - E_- = \underline{\underline{0}}.$$

<u>5. The Force on Charges in Electric Fields</u>



The force on a single charge, as shown at the right, can now be written in terms of the field it feels. According to the definition of electric field, $\vec{E} \equiv \frac{\vec{F}_e}{q}$ $\quad \vec{F}_e = q\vec{E}$.

<u>*Example 10:*</u> *The Earth has an electric field of about 150N/C pointed downward. A 1.00μm radius water droplet is suspended in calm air. Find (a)the mass of the water droplet, (b)the charge on the water droplet and (c)the number of excess electrons on the droplet.*



(a)Use the definition of density and the volume of a sphere,

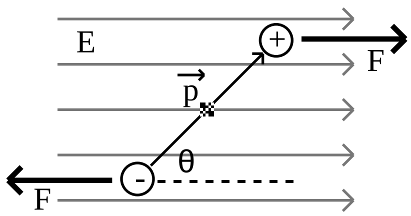
$$\rho \equiv \frac{m}{vol} \qquad m = \rho(vol) = \rho\tfrac{4}{3}\pi r^3 = (1000)\tfrac{4}{3}\pi(1.00\text{x}10^{-6})^3 = \underline{\underline{4.19\text{x}10^{-15}\text{kg}}}$$

(b)The forces on the droplet are its weight and the electric force. Using Newton's Second Law, $\Sigma F = ma \qquad F_e - F_g = 0 \qquad F_e = F_g$

Using the definitions of the electric and gravitational fields,

$$qE = mg \qquad q = \frac{mg}{E} = \frac{(4.19\text{x}10^{-15})(9.80)}{150} = \underline{\underline{2.74\text{x}10^{-16}\text{C}}}$$

(c)Since charge is quantized, $q = Ne \qquad N = \frac{q}{e} = \frac{2.74\text{x}10^{-16}}{1.60\text{x}10^{-19}} = \underline{\underline{1710 \text{ electrons}}}$



A dipole in a constant electric field as shown at the left. It feels no net force because the two forces on it caused by the field are equal and opposite. The dipole does feel a net torque, however. This torque tends to align it with the field.

$$\sum \tau = (a\sin\theta)(qE) + (a\sin\theta)(qE) = 2aqE\sin\theta = pE\sin\theta$$

This torque points into the paper so we can write the torque on the dipole as,

$$\boxed{\vec{\tau} = \vec{p} \times \vec{E} \quad \text{The Torque on a Dipole}}$$

The potential energy of the dipole can be found from the definition of potential energy, $\Delta U \equiv -W_c$.

The work done as the dipole rotates through an angle, $d\theta$, is, $dW = \tau\, d\theta = -pE\sin\theta\, d\theta$.

The total work done as the angle goes from $\pi/2$ to θ is, $W = -\int_{\pi/2}^{\theta} pE\sin\theta\, d\theta = pE\cos\theta$.

The potential energy is $\Delta U = U(\theta) - U(\pi/2) = -pE\cos\theta \quad U = -\vec{p} \cdot \vec{E}$, where the zero for potential energy is $\theta = \pi/2$.

$$\boxed{U = -\vec{p} \cdot \vec{E} \quad \text{The Potential Energy of a Dipole}}$$

<u>*Example 11:*</u> *Water molecules have a dipole moment of 6.20x10^{-30} C·m. Find (a)the maximum torque on a water molecule in the E-field of Earth and (b)the potential energy lost as the molecule moves from the position of maximum torque until it aligns with the field.*

(a)The torque on a dipole in a constant field is $\vec{\tau} = \vec{p} \times \vec{E}$. The maximum occurs when the dipole moment is perpendicular to the field, $\tau = pE = (6.20\text{x}10^{-30})(150) = \underline{\underline{9.30\text{x}10^{-28}\,\text{N m}}}$

(b)The potential energy of a dipole is $U = -\vec{p} \cdot \vec{E}$. When the moment is perpendicular to the field this is zero. When the moment is aligned with the field $U = -pE = -9.30\text{x}10^{-28}\,\text{J}$. This then is the energy that is lost $U_{lost} = \underline{\underline{9.30\text{x}10^{-28}\,\text{J}}}$.

<u>Chapter 23 - Summary</u>

The Definition of Electric Field $\vec{E} \equiv \dfrac{\vec{F}}{q}$

Drawing Field Lines

The Electric Field due to a Point Charge $\vec{E} = k\dfrac{q}{r^2}\hat{r}$

The Electric Field due to a Continuous Charge Distribution $\vec{E} = k\int \dfrac{dq}{r^2}\hat{r}$

The Definition of Electric Dipole Moment $\vec{p} \equiv q\vec{d}$

The Torque on a Dipole $\vec{\tau} = \vec{p} \times \vec{E}$

The Potential Energy of a Dipole $U = -\vec{p} \cdot \vec{E}$