Chapter 26 - Capacitance

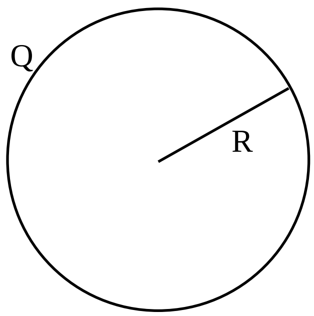
Problem Set #5 - due:
Ch 26 - 2, 3, 5, 7, 9, 15, 22, 26, 29, 61, 63, 64

The ideas of energy storage in E-fields can be carried a step further by understanding the concept of "Capacitance."

Lecture Outline

1. The Definition of Capacitance
2. Capacitors in Circuits
3. Energy Storage in Capacitors and Electric Fields
4. Dielectrics in Capacitors

1. The Definition of Capacitance

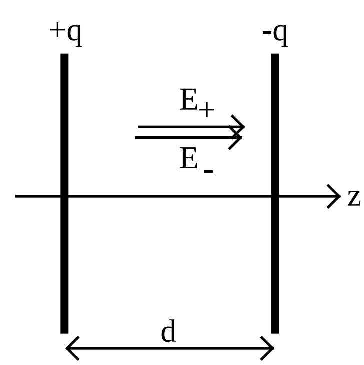


Consider a sphere with a total charge, Q, and a radius, R. From previous problems we know that the potential at the surface is, $V = k\frac{Q}{R}$. Putting more charge on the sphere stores more energy, but the ratio of energy or potential to charge depends only on R, not on Q or V. That is, $\frac{Q}{V} = \frac{R}{k}$. It's true for all charged objects that the ratio of potential to voltage depends only on the shape, so this ratio is defined as the capacitance.

$$\boxed{C \equiv \frac{Q}{V} \quad \text{The Defintion of Capacitance}}$$

The units of capacitance are $\frac{1\text{ Coulomb}}{1\text{ Volt}} \equiv 1\text{ Farad} \equiv 1\text{ F}$.

Example 1: Calculate the capacitance of two equal but oppositely charged plates of area, A, and separation, d. Neglect any edge effects.



The potential difference between the plates is, $\Delta V = -\int \vec{E} \cdot d\vec{s}$.

The field between the plates is just the sum of the fields due to the individual plates (see Ch 23 - example 9),

$$\vec{E} = \vec{E}_+ + \vec{E}_- = \frac{\sigma}{2\varepsilon_o}\hat{k} + \frac{\sigma}{2\varepsilon_o}\hat{k} = \frac{\sigma}{\varepsilon_o}\hat{k} = \frac{q}{\varepsilon_o A}\hat{k}.$$

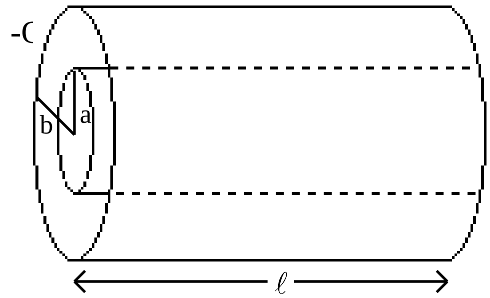
Using $d\vec{s} = dz\,\hat{k}$, the voltage on the capacitor can be written as,

$$\Delta V = -\int \frac{q}{\varepsilon_o A}\hat{k} \cdot dz\,\hat{k} = -\int \frac{q}{\varepsilon_o A}dz = -\frac{q}{\varepsilon_o A}\Delta z \qquad V = \frac{qd}{\varepsilon_o A}.$$

Applying the definition of capacitance,

$$C \equiv \frac{Q}{V} = \frac{q}{\frac{qd}{\varepsilon_o A}} \qquad C = \varepsilon_o \frac{A}{d}.$$ Note that the capacitance only depends on the shape.

<u>Example 2:</u> Find the capacitance of two concentric cylindrical conductors of radii a and b with a length, ℓ. Show that the result is consistent with example 1.



Assume the cylinders have equal and opposite charges, Q. Then the potential difference between them is,

$$\Delta V = -\int \vec{E} \cdot d\vec{s}$$

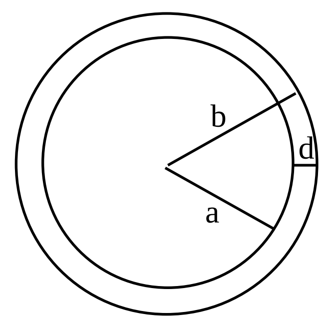
where $\vec{E} = \frac{2k\lambda}{r}\hat{r} = \frac{2kQ}{r\ell}\hat{r}$ from example 7 of chapter 24.

Using $d\vec{s} = dr\,\hat{r}$, the voltage on the capacitor can be written as,

$$\Delta V = -\int \frac{2kQ}{r\ell}\hat{r} \cdot dr\,\hat{r} = -\frac{2kQ}{\ell}\int_a^b \frac{1}{r}dr = -\frac{2kQ}{\ell}\ln\frac{b}{a} \qquad V = \frac{2kQ}{\ell}\ln\frac{b}{a}.$$

Using the definition of capacitance, $C \equiv \frac{Q}{V} = \frac{Q}{\frac{2kQ}{\ell}\ln\frac{b}{a}} \qquad C = \frac{\ell}{2k\ln\frac{b}{a}}$.

Again the result depends only on geometry.



When a and b get very large the concentric cylinders look like parallel plates. The distance between the plates is d = b - a. In terms of a and d, $\frac{b}{a} = \frac{a+d}{a} = 1 + \frac{d}{a}$.

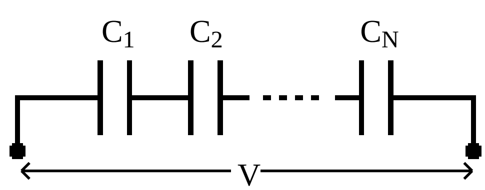
Now the capacitance can be written, $C = \frac{\ell}{2k\ln\left(1+\frac{d}{a}\right)}$.

In the limit $a \to \infty$ and d is small, the Taylor expansion of the logarithm can be used, $\ln(1+\delta) \approx \delta\left(1-\frac{\delta}{2}\right) \qquad \ln\left(1+\frac{d}{a}\right) \approx \frac{d}{a}\left(1-\frac{d}{2a}\right) \approx \frac{d}{a}$.

In this limit, $C = \frac{\ell}{2k\frac{d}{a}} = \frac{4\pi\varepsilon_o \ell a}{2d} = \varepsilon_o\frac{2\pi a\ell}{d} = \varepsilon_o\frac{A}{d}$ as expected.

<u>2. Capacitors in Circuits</u>

Bring some capacitors



<u>Capacitors in Series:</u>

By the Law of Conservation of Energy the sum of the voltages on the capacitors must equal the applied voltage.

$$V = V_1 + V_2 + \cdots + V_N$$

Using the definition of capacitance,

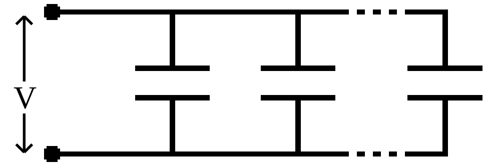
$$\frac{Q}{C_s} = \frac{Q_1}{C_1} + \frac{Q_2}{C_2} + \cdots + \frac{Q_N}{C_N}$$

The Law of Conservation of Charge requires all the charges to be equal, $Q = Q_1 = Q_2 = \cdots = Q_N$.

$$\frac{Q}{C_s} = \frac{Q}{C_1} + \frac{Q}{C_2} + \cdots + \frac{Q}{C_N} \qquad \frac{1}{C_s} = \frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_N} \qquad \frac{1}{C_s} = \sum \frac{1}{C_i}$$

$$\boxed{\frac{1}{C_s} = \sum \frac{1}{C_i} \quad \text{The Addition of Capacitors in Series}}$$

Capacitors in Parallel:



By the Law of Conservation of Charge the sum of the charges on the capacitors must equal the supplied charge.

$$Q = Q_1 + Q_2 + \cdots + Q_N$$

Using the definition of capacitance,

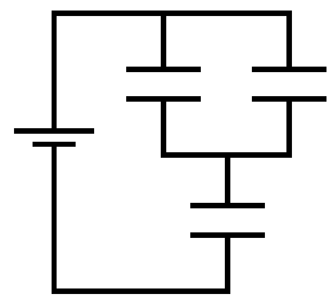
$$C_p V = C_1 V_1 + C_2 V_2 + \cdots + C_N V_N$$

The Law of Conservation of Energy requires all the voltages to be equal, $V = V_1 = V_2 = \cdots = V_N$.

$$C_p V = C_1 V + C_2 V + \cdots + C_N V \qquad C_p = C_1 + C_2 + \cdots + C_N \qquad C_p = \sum C_i$$

$$\boxed{C_p = \sum C_i \quad \text{The Addition of Capacitors in Parallel}}$$

Example 3: Find (a)the equivalent capacitance, (b)the charge on each capacitor and (c)the potential difference for each capacitor in the circuit shown. Given V=1.50V, C_1=4.00μF, C_2=8.00μF, and C_3=6.00μF.



(a)C_1 and C_2 are in parallel so $C_p = C_1 + C_2 = 12.0\,\mu F$.

Now we can imagine C_p in series with C_3 giving a total capacitance of

$$\frac{1}{C_s} = \frac{1}{C_p} + \frac{1}{C_3} \qquad C = \frac{C_p C_3}{C_p + C_3} = 4.00\,\mu F$$

(b)&(c)Using the definition of capacitance we can find the total charge $Q = CV = (4\mu F)(1.5V) = 6.00\,\mu C$.

This must equal the charge on C_3 by the Law of Conservation of Charge, Q_3=6.00μC.

Now the voltage on C_3 must be $C \equiv \frac{Q}{V} \qquad V_3 = \frac{Q_3}{C_3} = \frac{6\mu C}{6\mu F} = 1V$. According to the Law of Conservation of Energy that leaves 0.5V on C_1 and C_2. Now the charge on C_1 is,

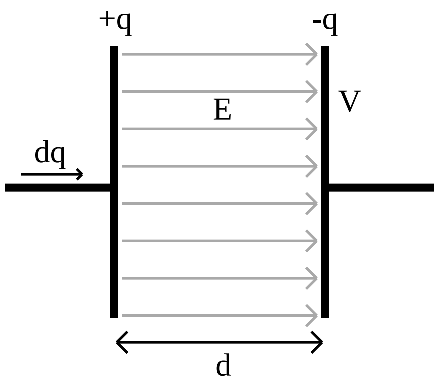
$Q_1 = C_1 V_1 = (4\mu F)(0.5V) = 2.00\,\mu C$ and the charge on C_2 is,

$Q_2 = C_2 V_2 = (8\mu F)(0.5V) = 4.00\,\mu C$. Note that $Q_3 = Q_1 + Q_2$.

In summary,

Q(μC)	C(μF)	V(V)
2.00	4.00	0.500
4.00	8.00	0.500
6.00	6.00	1.00

3. Energy Storage in Capacitors and Electric Fields



Suppose we are trying to put a total charge Q on a capacitor. How much energy will it take? Assume at some point the charge on the capacitor is q and the potential difference is V, using the electric potential energy we can find the energy needed to add a small amount of charge, dq

$$\Delta U = q\Delta V \qquad dU = dq\, V.$$

Using the definition of capacitance

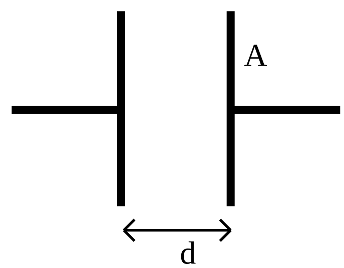
$$dU = dq\,\frac{q}{C}.$$

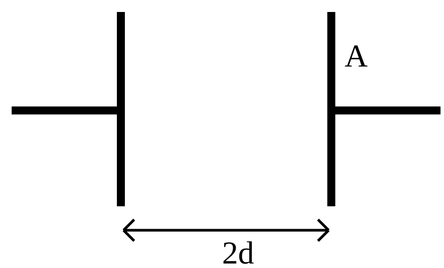
To find the total energy to charge the capacitor from q=0 to q=Q integrate,

$$\int_0^U dU = \int_0^Q dq\,\frac{q}{C} \qquad U = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}C\,V^2.$$

$$\boxed{U = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}C\,V^2 \quad \text{Stored Energy in Capacitors}}$$

<u>*Example 4:*</u> *A parallel plate capacitor of area, A, and plate separation, d, remains connected to a battery as the plates are pulled apart until they are separated by 2d. Find the change in stored energy.*





The initial stored energy is

$$U_o = \frac{1}{2}C_o\,V_o^2$$

where C_o is the initial capacitance, $C_o = \varepsilon_o\frac{A}{d}$, and V_o is the initial potential difference due to the battery. The final potential energy is

$$U = \frac{1}{2}C\,V_o^2$$

where C is the final capacitance, $C = \varepsilon_o\frac{A}{2d}$. The battery keeps the potential difference constant. The change in stored energy is,

$$\Delta U \equiv U - U_o = \frac{1}{2}C\,V_o^2 - \frac{1}{2}C_o\,V_o^2 = \frac{1}{2}\varepsilon_o\frac{A}{2d}\,V_o^2 - \frac{1}{2}\varepsilon_o\frac{A}{d}\,V_o^2 = -\frac{1}{4}\varepsilon_o\frac{A}{d}V_o^2$$

How can the energy drop when work is done to pull the plates apart ?

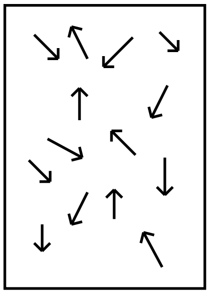
We can attribute this energy to the field, instead of to the capacitor. Using the capacitance of parallel plates and the relationship between the field and the potential,

$U = \frac{1}{2}C\,V^2 = \frac{1}{2}\left(\varepsilon_o\frac{A}{d}\right)(E\ d)^2 = \frac{1}{2}\varepsilon_o E^2 Ad = \frac{1}{2}\varepsilon_o E^2 \mathit{vol}$. In terms of the energy density,

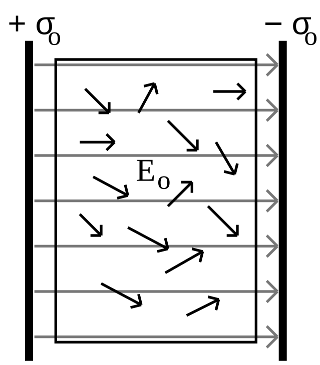
$$\boxed{u \equiv \frac{U}{\mathit{vol}} = \tfrac{1}{2}\varepsilon_o E^2 \quad \text{Stored Energy in Electric Fields}}$$

This expression turns out to be true for all E-fields.

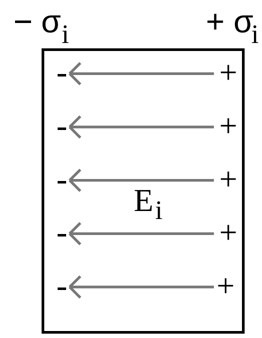
4. Dielectrics in Capacitors



Dielectrics are insulators. Electrons are not free to flow from one molecule to another. The atoms in a dielectric can have dipole moments. In a typical chunk of dielectric material these dipoles are randomly aligned and therefor produce no net field as shown.



When a dielectric is placed between the plates of a capacitor with a surface charge density σ_o the resulting electric field, E_o, tends to align the dipoles with the field. This results in a net charge density σ_i induced on the surfaces of the dielectric which in turns creates an induced electric field, E_i, in the opposite direction to the applied field. The total field inside the dielectric is reduced to,



$$E = E_o - E_i$$

The dielectric constant is defined as the ratio of the applied field to the total field, $\kappa \equiv \dfrac{E_o}{E}$. Substituting for E and solving for the induced field, $\kappa \equiv \dfrac{E_o}{E_o - E_i}$ $\quad E_i = \left(1 - \frac{1}{\kappa}\right)E_o$. Note that $\kappa=1$ is a perfect insulator such as a vacuum and $\kappa=\infty$ is a perfect conductor.

How does the introduction of a dielectric affect the capacitance of a capacitor? Recall the calculation of the capacitance of parallel plates starts with the calculation of the potential difference,

$\Delta V = -\int \vec{E} \cdot d\vec{s} = -\int \dfrac{\vec{E}_o}{\kappa} \cdot d\vec{s} = -\frac{1}{\kappa}\int \vec{E}_o \cdot d\vec{s} = \frac{1}{\kappa}\Delta V_o$. The potential difference will be smaller by a factor of κ.

Applying the definition of capacitance, $C \equiv \dfrac{Q}{V} = \dfrac{Q}{\frac{1}{\kappa}V_o} = \kappa\dfrac{Q}{V_o} = \kappa C_o$. The capacitance is larger by a factor of κ.

$$\boxed{C = \kappa C_o \quad \text{Capacitors with Dielectrics}}$$

<u>Example 5:</u> A parallel plate capacitor with 100cm² area and 2.00mm plate separation is connected to a 10.0V battery. Find the capacitance, charge, electric field and stored energy before and after it is disconnected from the battery and placed in oil of dielectric constant 5.00.

Using the capacitance of parallel plates, $C_o = \varepsilon_o \frac{A}{d} = \left(8.85\text{x}10^{-12}\right)\frac{0.0100}{.00222} = \underline{\underline{44.3\text{pF}}}$.

Using the definition of capacitance, $C_o \equiv \frac{Q_o}{V}$ $\quad Q_o = C_o V = \underline{\underline{443\text{pC}}}$.

The field can be found from the voltage, $E_o = -\frac{\partial V_o}{\partial x} = \frac{V_o}{d} = \frac{10.0}{0.00200} = \underline{\underline{5000\frac{\text{V}}{\text{m}}}}$.

The energy in a capacitor is, $U_o = \frac{1}{2}\frac{Q_o^2}{C_o} = \frac{1}{2}\frac{\left(443\text{x}10^{-12}\right)^2}{44.3\text{x}10^{-12}} = \underline{\underline{2.21\text{x}10^{-9}\text{J}}}$.

The new capacitance with the dielectric is, $C = \kappa C_o = 5.00 \;\; 44.3 = \underline{\underline{221\text{pF}}}$.

The charge remains the same because the battery is disconnected, $Q = Q_o = \underline{\underline{443\text{pC}}}$.

The new field is smaller by a factor of κ, $E = \frac{E_o}{\kappa} = \frac{5000}{5.00} = \underline{\underline{1000\frac{\text{V}}{\text{m}}}}$.

The new energy is, $U = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}\frac{\left(443\text{x}10^{-12}\right)^2}{221\text{x}10^{-12}} = \underline{\underline{0.443\text{x}10^{-9}\text{J}}}$. Where does the energy go?

<u>Example 6:</u> Repeat example 5 assuming the battery remains connected.

The new capacitance with the dielectric is still, $C = \kappa C_o = 5.00 \;\; 44.3 = \underline{\underline{221\text{pF}}}$.

This time the voltage remains the same but the charge changes,
$Q = CV = (221) \;\; (10.0) = \underline{\underline{2210\text{pC}}}$.

Since the voltage is the same, the field must be the same, $E = E_o = \underline{\underline{5000\frac{\text{V}}{\text{m}}}}$.

The new energy is, $U = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}\frac{\left(2210\text{x}10^{-12}\right)^2}{221\text{x}10^{-12}} = \underline{\underline{11.1\text{x}10^{-9}\text{J}}}$.

Where does the energy come from?

<u>Chapter 26 - Summary</u>

The Definition of Capacitance $C \equiv \frac{Q}{V}$

The Capacitance of Parallel Plates $C = \varepsilon_o \frac{A}{d}$

The Addition of Capacitors in Series $\frac{1}{C_s} = \sum \frac{1}{C_i}$

The Addition of Capacitors in Parallel $C_p = \sum C_i$

Stored Energy in Capacitors $U = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2} C V^2$

Stored Energy Electric Fields $u \equiv \frac{U}{vol} = \frac{1}{2}\varepsilon_o E^2$

Capacitors with Dielectrics $C = \kappa C_o$