The sum in problem 6.4a

The sum we need to complete is,

$$S=\sum_{m\neq n}\left(\frac{1}{n^2-m^2}\right),$$

where both n and m must be odd. This can be simplified using partial fractions,

$$S=\frac{1}{2n}\sum_{m\neq n}\left(\frac{1}{n-m}+\frac{1}{n+m}\right)=\frac{1}{2n}\left(\sum_{m\neq n}\frac{1}{n-m}+\sum_{m\neq n}\frac{1}{n+m}\right).$$

The sums can each be broken into the part up to n-1 and the part from n+1 upward,

$$S=\frac{1}{2n}\left(\sum_{m=0}^{n-1}\frac{1}{n-m}+\sum_{m=n+1}^{k}\frac{1}{n-m}+\sum_{m=0}^{n-1}\frac{1}{n+m}+\sum_{m=n+1}^{k}\frac{1}{n+m}\right),$$

where k will be allowed to approach infinity shortly. Reordering the sums,

$$S=\frac{1}{2n}\left(\sum_{m=0}^{n-1}\frac{1}{n-m}+\sum_{m=0}^{n-1}\frac{1}{n+m}+\sum_{m=n+1}^{k}\frac{1}{n+m}+\sum_{m=n+1}^{k}\frac{1}{n-m}\right).$$

Replacing m with –m in the first sum then flipping the limits,

$$S=\frac{1}{2n}\left(\sum_{m=1-n}^{0}\frac{1}{n+m}+\sum_{m=0}^{n-1}\frac{1}{n+m}+\sum_{m=n+1}^{k}\frac{1}{n+m}+\sum_{m=n+1}^{k}\frac{1}{n-m}\right)$$

Now, the first three sums can be turned into a single sum. You might be concerned about the redundant m=0 terms, but recall that m must be odd so those terms don't exist. You do however need to add in m=n term ($\frac{1}{2n}$) to write the single sum, so it must be subtracted out.

$$S=\frac{1}{2n}\left(-\frac{1}{2n}+\sum_{m=1-n}^{k}\frac{1}{n+m}+\sum_{m=n+1}^{k}\frac{1}{n-m}\right).$$

Now, replace m with j-n in the first sum,

$$S=\frac{1}{2n}\left(-\frac{1}{2n}+\sum_{j=1}^{k+n}\frac{1}{j}+\sum_{m=n+1}^{k}\frac{1}{n-m}\right).$$

Replace m with n+j in the last sum,

$$S=\frac{1}{2n}\left(-\frac{1}{2n}+\sum_{j=1}^{k+n}\frac{1}{j}-\sum_{j=1}^{k-n}\frac{1}{j}\right).$$

As k goes to infinity, the n can be ignored and the last two sums cancel leaving,

$$S=\frac{1}{2n}\left(-\frac{1}{2n}\right)=-\frac{1}{4n^2}.$$

Finally,

$$\boxed{\sum_{m\neq n}\left(\frac{1}{n^2-m^2}\right)=-\frac{1}{4n^2}}.$$