Physics 435B - Partial Wave Analysis

The quantum scattering problem must have solutions of the form given in eq. 11.12,

$$\psi(r,\theta) \approx A\left\{e^{ikz} + f(\theta)\frac{e^{ikr}}{r}\right\}.$$

It is our job to write this as a linear combination of the eigenfunctions of Schrodinger's Equation for a spherically symmetric potential.

1. Begin by writing Schrodinger's Equation in spherical coordinates in a region far enough away from the scattering center that the potential is zero. Since this is a free particle, the eigenvalues are known to be,

$$E = \frac{\hbar^2 k^2}{2m}.$$

2. Separate out the angular parts by using,

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) - \frac{1}{\hbar^2 r^2}\hat{L}^2 \quad \text{and} \quad \psi(r,\theta,\phi) = R(r)Y_\ell^m(\theta,\phi).$$

Use the fact that $\hat{L}^2 Y_\ell^m(\theta,\phi) = \ell(\ell+1)\hbar^2 Y_\ell^m(\theta,\phi)$, and the substitution ρ = kr to show that the radial equation reduces to Bessel's Equation,

$$\rho^2 R'' + 2\rho R' + \left[\rho^2 - \ell(\ell+1)\right]R = 0.$$

3. The solutions to this version of Bessel's Equation are known as the Spherical Bessel Functions,

$$j_\ell(\rho) = \rho^\ell\left(-\frac{1}{\rho}\cdot\frac{d}{d\rho}\right)^\ell\left(\frac{\sin\rho}{\rho}\right) \quad \text{and} \quad n_\ell(\rho) = -\rho^\ell\left(-\frac{1}{\rho}\cdot\frac{d}{d\rho}\right)^\ell\left(\frac{\cos\rho}{\rho}\right).$$

Write a general solution to the radial equation.

4. The azimuthal symmetry means that only spherical harmonics with m=0 need to be used in the sum. Use eq. 11.22,

$$Y_\ell^0(\theta,\phi) = \sqrt{\frac{2\ell+1}{4\pi}}P_\ell(\cos\theta)$$

to rewrite the general solution to Schrodinger's Equation as a sum over the Legendre polynomials.

5. Now, let's get to the first chore which is to write e^{ikz} as a linear combination of the spherical polar eigenfunctions. Keep in mind that e^{ikz} is finite at the origin where the n's are infinite.

6. Show that the expansion coefficient, α_ℓ, is given by,

$$\alpha_\ell j_\ell(\rho) = \sqrt{\pi(2\ell+1)}\int_{-1}^{+1} e^{i\rho\eta}P_\ell(\eta)\,d\eta$$

where $\eta = \cos\theta$. Use the orthogonality of the associated Legendre polynomials,

$$\int_{-1}^{+1} P_{\ell'}(\eta)P_\ell(\eta)\,d\eta = \frac{2}{2\ell+1}\delta_{\ell'\ell}.$$

7. Use the definition of the associated Legendre polynomials from eq. 4.28,

$$P_\ell(\eta) = \frac{1}{2^\ell \ell!}\frac{d^\ell}{d\eta^\ell}\left(\eta^2 - 1\right)^\ell$$

and perform ℓ integrations by parts. Use your result along with an alternative definition of the Spherical Bessel Function,

$$j_\ell(\rho) = \frac{\rho^\ell}{2^{\ell+1}\cdot \ell!}\int_{-1}^{+1} e^{i\rho s}\left(1 - s^2\right)^\ell ds$$

to show that

$$\alpha_\ell = i^\ell\sqrt{4\pi(2\ell+1)}\,.$$

You should now be able to write Rayleigh's formula (eq. 11.28).

8. Now, on to the $\frac{e^{ikr}}{r}$ term. Note that it is infinite at the origin so the n's must be included. You could use Fourier's trick, but let's be smarter and use the fact that

$$j_\ell(\rho) \approx \frac{\sin\left(\rho - \ell\frac{\pi}{2}\right)}{\rho} \quad \text{and} \quad n_\ell(\rho) \approx -\frac{\cos\left(\rho - \ell\frac{\pi}{2}\right)}{\rho} \quad \text{for } \rho \gg 1,$$

to write the radial portion of the wave function for large r as a linear combination of the radial eigenfunctions. Use the definition of the spherical Hankel functions of eq. 11.19 to simplify the result.

9. Using the solution of Schrodinger's Equation from problem 4, write the scattered wave term as a sum over spherical harmonics with an expansion coefficient $C_\ell \equiv \sqrt{4\pi(2\ell+1)}a_\ell$. You should be able to get from the previous result to equation 11.29.