The Zeeman Effect - Calculating Splittings and Relative Intensities

The spectrum of mercury has a very strong green line. The green line has a wavelength of 546.1nm and is known to be a $^3S_1 \rightarrow {}^3P_2$ transition. The purpose of this exercise is to calculate the energy splitting of the green line and also the relative intensities of the "anomalous" Zeeman lines. Please take this exercise in the spirit of a summary of the course to this point. This is your chance to be sure that you understand what we've done.

(a)Write the Hamiltonian for a mercury atom. Now, replace $\vec{p}$ with $\boxed{\vec{p} - q\vec{A} = \vec{p} + e\vec{A}}$ to derive the portion of the Zeeman term due to the electron's orbital motion assuming a weak and constant magnetic field along the z-direction. You'll need to require that $\nabla \bullet \vec{A} = 0$ which is called the "Coulomb Gauge."

(b)Show that $\vec{A} = \frac{1}{2}(B\hat{k} \times \vec{r})$ is consistent with a constant field along z. Rewrite the Hamiltonian in terms of $\vec{B} = B\hat{k}$, the total orbital angular momentum $\vec{L}$, and the Bohr Magneton $\mu_B \equiv \frac{e\hbar}{2m}$.

(c)Add the term due to the electron's spin interacting with the magnetic field. Now find the first order energy corrections to the eigenstates of total angular momentum, $\left|n,\ell,s,j,m_j\right\rangle$. Express Zeeman correction in terms of the Lande' g-factor, g_J. (see Griffith's p. 278).

(d) Find the energy splittings associated with the $^3S_1$ and $^3P_2$ states. Draw a diagram of these two energy levels as a function of B showing the splittings. Label the states with their m_j values.

(e)Start with eq. 9.31 and replace $\hat{k}$ with $\hat{\varepsilon}$, a unit vector along the polarization direction, to show that the spontaneous emission rate between the $^3S_1$ and $^3P_2$ states is proportional to

$$R \propto \left|\left\langle n,\ell,s,j,m\right|\vec{r} \bullet \hat{\varepsilon}\left|n',\ell',s',j',m'\right\rangle\right|^2.$$

(f) Show that for light polarized parallel to the field, $R_\pi \propto \left|\left\langle n,\ell,s,j,m\right|z\left|n',\ell',s',j',m'\right\rangle\right|^2$ and for light polarized perpendicular to the field (say the x-axis) $R_\sigma \propto \left|\left\langle n,\ell,s,j,m\right|x\left|n',\ell',s',j',m'\right\rangle\right|^2$.

(g)Express z and x in terms of the spherical harmonics to show that,

$$z = \sqrt{2} f(r)\left|1,0\right\rangle \text{ and } x = f(r)\left\{\left|1,-1\right\rangle - \left|1,1\right\rangle\right\} \text{ where } f(r) = \sqrt{\tfrac{2\pi}{3}}\, r \text{ and } Y_\ell^m \equiv \left|\ell,m\right\rangle$$

Substitute these expressions into R_π and R_σ. Separate the radial and angular parts of R_π and R_σ. The radial part is the same for both. Show that,

$$R_\pi \propto 2I\left|\left\langle j,m\right|\left|1,0\right\rangle\left|j',m'\right\rangle\right|^2 \text{ and } R_\sigma \propto I\left|\left\langle j,m\right|\left\{\left|1,-1\right\rangle\left|j',m'\right\rangle - \left|1,1\right\rangle\left|j',m'\right\rangle\right\}\right|^2 \text{ where } I = \left|\left\langle n,\ell,s\right|f(r)\left|n',\ell',s'\right\rangle\right|^2.$$

(h)Note that j' = 1 and m' = 0, ±1 for the $^3S_1$ state. Use table 4.7 to express the terms $\left|1,-1\right\rangle\left|1,m'\right\rangle$, $\left|1,0\right\rangle\left|1,m'\right\rangle$ and $\left|1,+1\right\rangle\left|1,m'\right\rangle$ as linear combinations of coupled angular momenta of the form $\left|1,1,2,m''\right\rangle$, $\left|1,1,1,m''\right\rangle$ and $\left|1,1,0,m''\right\rangle$ for each possible value of m'.

(i)Use the orthogonality of the angular momentum eigenfunctions to find the three allowed values of R_π and the six allowed values of R_σ. Notice that this puts restrictions on which initial states and final states can be involved in transitions. These restrictions are called "selection rules." State the selection rules.

(j)Draw the allowed transitions on an energy level diagram with the highest energy transitions on the left and the lowest energy transitions on the right. Directly below the energy level diagram sketch the intensity versus position on a CCD. Sketch the relative intensities to scale.