Beats in an Oscillator Near Resonance

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The sinusoidally driven mass-spring system exhibits a classic example of resonance. As the frequency of the driver (ω_D) approaches the natural frequency of the mass-spring system (ω_0), the amplitude of the mass oscillations grows until it reaches a maximum when the two frequencies are equal. While the actual analysis of this situation using Newton's laws is somewhat complicated, most students are reasonably comfortable with the conceptual explanation: the response of the system is greatest when you drive it at the frequency at which it naturally oscillates.

Surprising Beats

During a summer internship at PASCO scientific Corp., a student of ours (David Atkinson) developed a high school laboratory exercise to illustrate this resonance phenomena using just such a mass-spring system. During a presentation of his experimental results for ω_D near but *not equal* to ω_0, we noticed an interesting feature in the motion: beats. That is, the amplitude of the mass oscillations was not constant in time, but varied with a period much greater than the period of the oscillations themselves (Fig. 1). Since beats are produced by the superposition of two nearly equal frequencies, a quite reasonable guess for these frequencies would be ω_D and ω_0. While providing an excellent demonstration of beats in a mechanical system, is there really any more to be said about this situation? In fact, we were surprised by the beats' presence, since we initially rejected the possibility of the natural frequency being involved in their production.

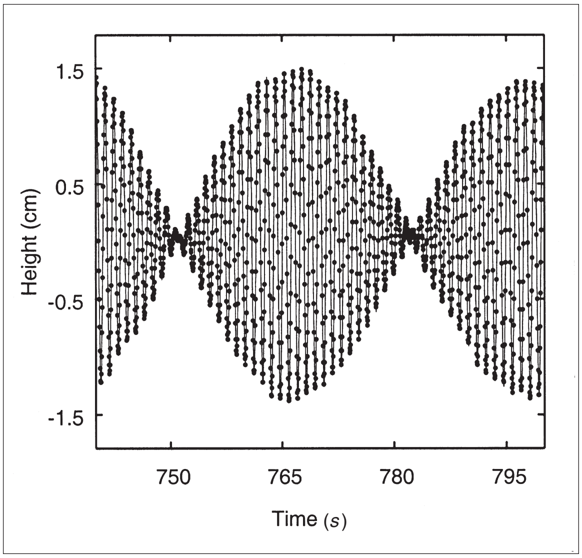

Fig. 1. Amplitude of the oscillations of a hanging mass-spring system driven by an audio speaker. The driving frequency is near resonance and the time is measured since the last slight frequency adjustment was made.

Why Be Surprised?

To understand why we were surprised, one must examine the oscillator in more detail.[1] For an *undamped*, *undriven* oscillator with mass m and spring constant k, Newton's laws predict that the position of the mass is given by

$$x_1 = A \sin(\omega_0 t), \tag{1}$$

where $\omega_0 = \sqrt{k/m}$. An oscillator given a sudden, brief displacement from equilibrium, i.e. given an initial "kick," will subsequently oscillate at ω_0. However, any real oscillator has damping that converts the mechanical energy to thermal energy, thus causing the amplitude to decay over time. The Newtonian analysis is a little more complicated in this case, but the results are not surprising:

Fig. 2. The mass-spring system with PASCO function generator and motion detector.

$$x_1 = Ae^{-t/\tau} \sin(\omega' t), \qquad (2)$$

where τ is the time constant for the amplitude decay and $\omega' = \sqrt{\omega_0^2 - (1/\tau)^2}$. The exact form of this solution assumes that the drag force is directly proportional to the velocity of the mass, but the general feature of decaying amplitude will exist independent of the precise nature of the drag. The important point is that the natural oscillations essentially stop after a time interval equal to several "decay constants"; hence, this type of motion is known as the "transient solution."

If one drives such an oscillator with a sinusoidal force, Newton's laws predict that the mass will oscillate at this driving frequency:

$$x_2 = D \sin(\omega_D t + \vartheta). \qquad (3)$$

The resonance phenomena is expressed mathematically in the fact that the amplitude D is a function of the driving frequency:

$$D = \frac{F_0 / m}{\sqrt{\omega_D^2 - \omega_0^2)^2 - (2\omega_D / \tau)^2}}\,. \qquad (4)$$

If the drag is small, then τ is large which causes D to peak sharply in the neighborhood of ω_0. The motion described by Eq. (3) does not decay over time and so is usually referred to as the "steady state solution." The complete motion of the mass is given by the sum $x = x_1 + x_2$. Obviously only the steady state part of the motion (x_2) will contribute after several time constants have elapsed.

We were surprised by the presence of the beats because they were observed as David slowly adjusted ω_D toward resonance, more than *one hour* after the driver was started and the system was given its initial "kick." We surmised that the transient oscillations at ω_0 would have been insignificant by this time, and so were led to conclude that the second frequency beating against the driving frequency must have arisen from some subtler source.

Simplicity Over Subtlety

The first task was to build the system that produced the beats and experimentally investigate more precisely what was occurring. As shown in Fig. 2, the system is quite basic: a mass is hung from a spring whose other end is connected to a small plastic hook glued to a 40-W speaker that is driven by a high-stability PASCO function generator. A PASCO ultrasonic motion detector monitors the motion. We were quickly able to confirm that beats were present in the vertical oscillation of the mass, and that they occurred for a small range of frequencies about the resonance frequency. In these measurements, our typical procedure was to adjust the frequency about resonance only after the oscillator had been driving the mass for at least 30 minutes. After observing beats at one frequency, we would change the frequency by one to three percent and observe beats at the new frequency. The beats shown in Fig. 1 were produced by this procedure, the data being taken 12 minutes after a slight detuning from resonance: $(\omega_D - \omega_0)/\omega_0 = 0.03$.

The reader may recognize in this procedure the source of the "mysterious" beats. While the beats were observed long after the driver was started, the changing of the frequency, even if only by a few percent, constitutes a breaking of the purely sinusoidal forcing function. Such a change provides a "kick" to the system and thus causes the reintroduction of the transient oscillations. Combining these gentle "kicks" with the quite large decay time for these oscillations, $\tau \approx 8$ min, it is quite easy to inadvertently produce *seemingly* steady state beats.

Since the beats *are* produced by the slight frequency difference between the steady state ω_D and the transient ω_0, they will disappear if the mass-spring system is left undisturbed, but one must wait more than 20 minutes for their extinction (Fig. 3). Indeed, if one wishes to use this system to map out the resonance curve, $D(\omega_D)$ versus ω_D, the time involved would be prohibitive in a three-hour lab period. However, it is a simple matter to increase the drag by attaching a cardboard "air sail" to the bottom of the mass, thus decreasing the time to reach steady state.

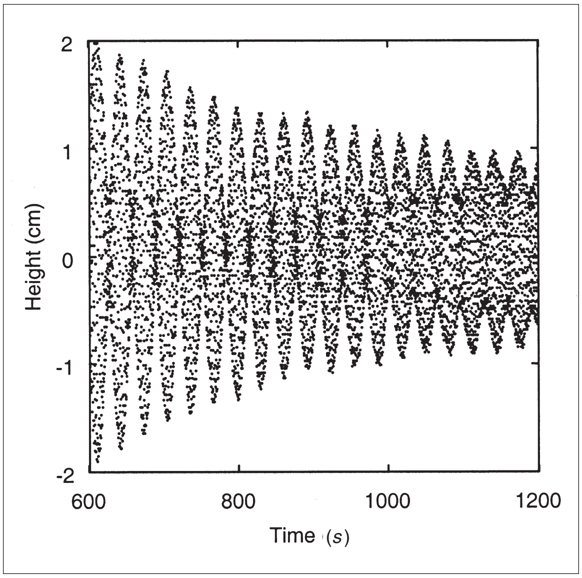

Fig. 3. Mass-spring oscillations shown on an expanded time scale. The slow damping of the beats is evident over the 10 minutes shown.